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Notes on a Generalization of the Stern-Gerlach Force

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In this paper I attempt to generalize the Stern-Gerlach force on an elementary particle to relativistic energies in a covariant manner. Of particular interest is the case of the longitudinal component of force on a particle which is longitudinally polarized. The force in this case is found to be proportional to γ; however, when integrating the energy increase through a TE rf cavity, it is found that the energy shift is roughly proportional to $1/\gamma$. For static magnetic gradients, such as from the ends of a solenoid, the energy increase from the gradient at one end of the solenoid is canceled by the opposite gradient at the other end. As a result this increased factor of γ in the force does not appear to be terribly useful. Alas, Maxwell and Einstein have conspired against us.

Preliminary Comments on Notation

For the this discussion I assume a flat Minkowski metric $g^{\alpha\beta}$ of the form

$$\begin{array}{cc} & \beta\rightarrow \\ \begin{array}{c}\alpha\\ \downarrow\end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{array} \tag{1}$$

The contravariant coordinates x^α in vector form are:

$$\mathrm{x} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{2}$$

The proper velocity u^α is normalized to the speed of light:

$$\mathrm{u} = \begin{pmatrix} \gamma c \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}. \tag{3}$$

The four-vector potential A^α has components

$$\mathrm{A} = \begin{pmatrix} \phi/c \\ A_x \\ A_y \\ A_z \end{pmatrix}. \tag{4}$$

The relativistic extension of the gradient operator in covariant form is

$$\partial^\alpha = \frac{\partial}{\partial x_\alpha}, \tag{5}$$

or in vector form

$$\begin{pmatrix} \frac{1}{c}\frac{\partial}{\partial t} \\ -\nabla \end{pmatrix}. \tag{6}$$

Another useful relation is the total derivative with respect to the proper time, τ:

$$\frac{\partial}{\partial \tau} = u_\beta \partial^\beta = u_\beta \frac{\partial}{\partial x_\beta} = \gamma \left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right). \tag{7}$$

The electromagnetic Faraday tensor is defined by

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{8}$$

and has components

$$\mathrm{F} = \begin{array}{cc} & \beta\rightarrow \\ \begin{matrix}\alpha \\ \downarrow\end{matrix} & \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \end{array} \tag{9}$$

Maxwell's equations may be written in covariant form as

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0, \quad \text{and} \quad \partial_\beta F^{\alpha\beta} = 4\pi J^\alpha, \tag{10}$$

where J is the covariant current density. The dual of the Faraday tensor

$$^*F^{\alpha\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \tag{11}$$

has the components

$$^*\mathrm{F} = \begin{array}{cc} & \beta\rightarrow \\ \begin{matrix}\alpha \\ \downarrow\end{matrix} & \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} \end{array} \tag{12}$$

The angular momentum density tensor is defined by

$$M^{\gamma\alpha\beta} = x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma}, \tag{13}$$

where $T^{\alpha\beta}$ is the Stress energy tensor.[†] The total angular momentum tensor is given by

$$\begin{aligned} J^{\alpha\beta} &= \int M^{0\alpha\beta}\, d^3x \\ &= \int (x^\alpha T^{\beta 0} - x^\beta T^{\alpha 0})\, d^3x. \end{aligned} \tag{14}$$

[†] See §§2.8 and 2.9 of Steven Weinberg, *Gravitation and Cosmology*, John Wiley & Sons (1972) for a nice covariant discussion of the stress-energy tensor and spin.

The spin or intrinsic angular momentum four vector may be obtained by

$$S^{\alpha} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta} J_{\beta\gamma} u_{\delta}. \tag{15}$$

Calculation of Stern-Gerlach Force by Boost

In order to calculate the Stern-Gerlach force for a moving particle with spin we first calculate the magnetic field in the rest system of the particle in terms of the electric and magnetic fields in the lab. The proper force is then calculated in the rest system and finally boosted back to the lab. While one may argue about what happens to a magnetic moment of a moving particle, this procedure eliminates such worries since the Lorentz transformation of forces is well understood.

The contravariant derivative in the rest system[†] may be written in terms of the laboratory system coordinates as

$$\frac{\partial}{\partial x^{\diamond\alpha}} = \frac{\partial}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x^{\diamond\alpha}} \tag{16}$$

which has components

$$\frac{\partial}{\partial t^{\diamond}} = \gamma\left(\frac{\partial}{\partial t} + \beta c \frac{\partial}{\partial z}\right) \tag{17a}$$

$$\frac{\partial x}{\partial x^{\diamond}} = \frac{\partial}{\partial x} \tag{17b}$$

$$\frac{\partial x}{\partial y^{\diamond}} = \frac{\partial}{\partial y} \tag{17c}$$

$$\frac{\partial}{\partial z^{\diamond}} = \gamma\left(\frac{\partial}{\partial z} + \frac{\beta}{c}\frac{\partial}{\partial t}\right) \tag{17d}$$

Written in terms of the fields in the laboratory, the components of the rest frame's magnetic field are

$$B^{\diamond}_{\parallel} = B_{\parallel} \tag{18a}$$

$$\vec{B}^{\diamond}_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{\vec{\beta}}{c} \times \vec{E}\right). \tag{18b}$$

The Stern-Gerlach force in the rest system may be written as

$$\begin{aligned} \vec{F}^{\diamond} &= (\mu^{\diamond} \cdot \nabla^{\diamond})\vec{B}^{\diamond} \\ &= \left[\mu^{\diamond}_x \frac{\partial}{\partial x} + \mu^{\diamond}_y \frac{\partial}{\partial y} + \gamma\mu^{\diamond}_z\left(\frac{\partial}{\partial z} + \frac{\beta}{c}\frac{\partial}{\partial t}\right)\right]\vec{B}^{\diamond}. \end{aligned} \tag{19}$$

with components

$$F^{\diamond}_{\parallel} = \left[\mu^{\diamond}_x \frac{\partial}{\partial x} + \mu^{\diamond}_y \frac{\partial}{\partial y} + \gamma\mu^{\diamond}_z\left(\frac{\partial}{\partial z} + \frac{\beta}{c}\frac{\partial}{\partial t}\right)\right] B_{\parallel} \tag{19a}$$

$$\vec{F}^{\diamond}_{\perp} = \left[\mu^{\diamond}_x \frac{\partial}{\partial x} + \mu^{\diamond}_y \frac{\partial}{\partial y} + \gamma\mu^{\diamond}_z\left(\frac{\partial}{\partial z} + \frac{\beta}{c}\frac{\partial}{\partial t}\right)\right]\gamma\left(\vec{B}_{\perp} - \frac{\vec{\beta}}{c} \times \vec{E}\right). \tag{19b}$$

[†] Here the superscript "◇" is used to specify values in the rest system of the particle.

The four-force in the rest system is

$$\mathcal{F}^{\diamond} = \begin{pmatrix} 0 \\ F_x^{\diamond} \\ F_y^{\diamond} \\ F_z^{\diamond} \end{pmatrix} \tag{20}$$

since the particle is not moving in the rest system ($\vec{v}^{\diamond} = 0$) and $\mathcal{F}^{\diamond 0} = \vec{v}^{\diamond} \cdot \vec{F}^{\diamond} = 0$. Boosting this back to the lab gives the proper force

$$\mathcal{F} = \frac{d\mathrm{p}}{d\tau} = \begin{pmatrix} \gamma\vec{\beta} \cdot \vec{\mathcal{F}}^{\diamond} \\ \mathcal{F}_x^{\diamond} \\ \mathcal{F}_y^{\diamond} \\ \gamma\mathcal{F}_z^{\diamond} \end{pmatrix}. \tag{21}$$

In the lab has the parallel and transverse components of the force are

$$\vec{F}_{\parallel} = \mu_x^{\diamond}\frac{\partial B_{\parallel}}{\partial x} + \mu_y^{\diamond}\frac{\partial B_{\parallel}}{\partial y} + \gamma\mu_z^{\diamond}\left(\frac{\partial B_{\parallel}}{\partial z} + \frac{\beta}{c}\frac{\partial B_{\parallel}}{\partial t}\right) \tag{21a}$$

$$\vec{F}_{\perp} = \mu_x^{\diamond}\frac{\partial}{\partial x} + \mu_y^{\diamond}\frac{\partial}{\partial y} + \gamma\mu_z^{\diamond}\left(\frac{\partial}{\partial z} + \frac{\beta}{c}\frac{\partial}{\partial t}\right)\left(\vec{B}_{\perp} - \frac{\vec{\beta}}{c} \times \vec{E}\right), \tag{21b}$$

which may be combined to give

$$\begin{aligned}\vec{F} = {}& (\vec{\mu}^{\diamond} \cdot \nabla)\vec{B} + \frac{\gamma - 1}{\beta^2}(\vec{\beta} \cdot \vec{\mu}^{\diamond})(\vec{\beta} \cdot \nabla)\vec{B} + \frac{\gamma}{c}(\vec{\beta} \cdot \vec{\mu}^{\diamond})\frac{\partial \vec{B}}{\partial t} \\ & - \left[(\vec{\mu}^{\diamond} \cdot \nabla)\left(\frac{\vec{\beta}}{c} \times \vec{E}\right) + \frac{\gamma - 1}{\beta^2}(\vec{\beta} \cdot \vec{\mu}^{\diamond})(\vec{\beta} \cdot \nabla)\left(\frac{\vec{\beta}}{c} \times \vec{E}\right) + \frac{\gamma}{c}(\vec{\beta} \cdot \vec{\mu}^{\diamond})\left(\frac{\vec{\beta}}{c} \times \frac{\partial \vec{E}}{\partial t}\right)\right].\end{aligned} \tag{22}$$

For longitudinal polarization ($\vec{\mu}^{\diamond} = \mu^{\diamond}\hat{z}$) this gives

$$\vec{F} = \gamma\mu^{\diamond}\left(\frac{\partial \vec{B}}{\partial z} + \frac{\beta}{c}\frac{\partial \vec{B}}{\partial t}\right) - \gamma\mu^{\diamond}\frac{\beta}{c}\hat{z} \times \left(\frac{\partial \vec{E}}{\partial z} + \frac{\beta}{c}\frac{\partial \vec{E}}{\partial t}\right) \tag{23}$$

which is proportional to γ, whereas for transverse polarization ($\vec{\beta} \cdot \vec{\mu}^{\diamond} = 0$) the result simplifies to

$$\vec{F} = (\vec{\mu}^{\diamond} \cdot \nabla)\vec{B} - (\vec{\mu}^{\diamond} \cdot \nabla)\left(\frac{\vec{\beta}}{c} \times \vec{E}\right). \tag{24}$$

which is not proportional to γ.

Covariant Lagrangian for the Stern-Gerlach Force *

In order to construct a covariant Lagrangian for the Stern-Gerlach force we need to find a covariant expression which which reduces to the energy term $-\vec{B}\cdot\vec{S}$ in the rest system. Since there is a magnetic field in the expression we should expect to see the antisymmetric electromagnetic Faraday tensor or its dual in the expression. A term like $F^{\alpha\beta}S_\beta$ has an $\vec{E}\cdot\vec{S}$ term for its time-like component, whereas ${}^*F^{\alpha\beta}S_\beta$ has a $\vec{B}\cdot\vec{S}$ term. In the rest system the space-like components of the proper velocity are zero, so we might expect the interaction term to be proportional to $u_\alpha F^{\alpha\beta}S_\beta$. Since the magnetic moment in the rest system is $\vec{\mu}^\diamond = \frac{ge}{2m}\vec{S}^\diamond$, the expected Lagrangian should be

$$L(x,u;\tau) = \frac{1}{2}mu^\alpha u_\alpha + eA^\alpha u_\alpha + \frac{ge}{2mc}u_\alpha {}^*F^{\alpha\beta}S_\beta, \tag{25}$$

with the covariant canonical momentum components

$$\begin{aligned} P^\alpha &= \frac{\partial L}{\partial u_\alpha} \\ &= mu^\alpha + eA^\alpha + \frac{ge}{2mc}{}^*F^{\alpha\beta}S_\beta. \end{aligned} \tag{26}$$

To simplify some of the following algebra define

$$\mathcal{A}^\alpha = eA^\alpha + \frac{ge}{2mc}{}^*F^{\alpha\beta}S_\beta, \tag{27}$$

so that

$$L(x,u;\tau) = \frac{1}{2}mu^\alpha u_\alpha + \mathcal{A}^\alpha u_\alpha, \quad \text{and} \quad P^\alpha = mu^\alpha + \mathcal{A}. \tag{28}$$

Variation of the action between points π_1 and π_2

$$\delta I = \delta\int_{\pi_1}^{\pi_2} L\,d\tau = 0, \tag{29}$$

will yield the Euler equations of motion. Since $\delta u^\alpha = \frac{d}{d\tau}(\delta u^\alpha)$ the variation yields

$$\begin{aligned} \delta I &= \int_{\pi_1}^{\pi_2}\left\{(mu^\alpha + \mathcal{A}^\alpha)\frac{d}{d\tau}(\delta x_\alpha) + (\partial^\alpha\mathcal{A}^\beta)u_\beta\,\delta x_\alpha\right\}d\tau \\ &= [(mu^\alpha + \mathcal{A}^\alpha)\delta x_\alpha]|_{\pi_1}^{\pi_2} - \int_{\pi_1}^{\pi_2}\left\{-m\frac{du^\alpha}{d\tau} + (\partial^\alpha\mathcal{A}^\beta - \partial^\beta\mathcal{A}^\alpha)\frac{dx_\beta}{d\tau}\right\}\delta x_\alpha\,d\tau \end{aligned} \tag{30}$$

The leading expression in brackets is clearly zero since $\delta x^\alpha(\pi_1) = \delta x^\alpha(\pi_2) = 0$, therefor the part of the integrand inside the braces must be equal to zero, so

$$m\frac{du^\alpha}{d\tau} = u_\beta(\partial^\alpha\mathcal{A}^\beta - \partial^\beta\mathcal{A}^\alpha), \tag{31}$$

* This treatment of the relativistic Lagrangian follows W. K. H. Panofsky and M. Phillips *Classical Electricity and Magnetism*, Addison-Wesley (1962) but with the addition of spin.

or

$$m\frac{du^\alpha}{d\tau} = eF^{\alpha\beta}u_\beta + \frac{ge}{2mc}(\partial^\alpha {}^*\!F^{\beta\gamma} - \partial^\beta {}^*\!F^{\alpha\gamma})u_\beta S_\gamma. \tag{32}$$

A covariant super Hamiltonian may be constructed in the usual manner

$$\begin{aligned}\mathcal{H}(x,P;\tau) &= P^\alpha u_\alpha - L \\ &= \frac{1}{2m}\left[P^\alpha - eA^\alpha - \frac{ge}{2mc}{}^*\!F^{\alpha\beta}S_\beta\right]\left[P_\alpha - eA_\alpha - \frac{ge}{2mc}{}^*\!F_{\alpha\gamma}S^\gamma\right]\end{aligned} \tag{33}$$

One should note that

$$\mathcal{H} = \frac{1}{2}mu^\alpha u_\alpha = \frac{1}{2}mc^2, \tag{34}$$

which is a nicely conserved quantity.

Let us now evaluate the proper force components from the Euler equation. With no spin, we get the usual Lorentz force equation

$$\frac{d\vec{p}}{d\tau} = m\frac{d\vec{u}}{d\tau} = \gamma e(\vec{E} + \vec{v}\times\vec{B}) \tag{35}$$

The Stern-Gerlach force should come from the second part of the Euler equation. The summed product ${}^*\!F^{\beta\gamma}S_\gamma$ is

$$\begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}\begin{pmatrix}\vec{\beta}\cdot\vec{S} \\ -S_x \\ -S_y \\ -S_z\end{pmatrix} = \begin{pmatrix} -\vec{B}\cdot\vec{S} \\ -(\vec{\beta}\cdot\vec{S})\vec{B} - \frac{1}{c}\vec{E}\times\vec{S}\end{pmatrix}, \tag{36}$$

and

$$\begin{aligned}u_\beta {}^*\!F^{\beta\gamma}S_\gamma &= \gamma c\begin{pmatrix}1 & -\vec{\beta}\end{pmatrix}\begin{pmatrix} -\vec{B}\cdot\vec{S} \\ -(\vec{\beta}\cdot\vec{S})\vec{B} - \frac{1}{c}\vec{E}\times\vec{S}\end{pmatrix} \\ &= \gamma c\left[-\vec{B}\cdot\vec{S} + (\vec{\beta}\cdot\vec{S})(\vec{\beta}\cdot\vec{B}) + \frac{1}{c}\vec{\beta}\cdot(\vec{E}\times\vec{S})\right].\end{aligned} \tag{37}$$

The three component force may be written

$$\begin{aligned}m\frac{d\vec{u}}{d\tau} = \gamma\vec{F} &= \gamma e(\vec{E}+\vec{v}\times\vec{B}) \\ &+ \frac{ge}{2mc}\left\{-\gamma c\nabla\left[-\vec{B}\cdot\vec{S} + (\vec{\beta}\cdot\vec{S})(\vec{\beta}\cdot\vec{B}) + \frac{1}{c}\vec{\beta}\cdot(\vec{E}\times\vec{S})\right]\right. \\ &\left. - \gamma\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\left[-(\vec{\beta}\cdot\vec{S})\vec{B} - \frac{1}{c}\vec{E}\times\vec{S}\right]\right\}\end{aligned} \tag{38}$$

$$\begin{aligned}\vec{F} = e(\vec{E}+\vec{v}\times\vec{B}) &+ \frac{ge}{2m}\left\{\nabla(\vec{B}\cdot\vec{S}) - (\vec{\beta}\cdot\vec{S})\nabla(\vec{\beta}\cdot\vec{B}) - \frac{1}{c}\nabla[\vec{\beta}\cdot(\vec{E}\times\vec{S})]\right. \\ &\left. + \frac{1}{c}(\vec{\beta}\cdot\vec{S})\frac{\partial\vec{B}}{\partial t} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\times\vec{S} + (\vec{\beta}\cdot\vec{S})(\vec{\beta}\cdot\nabla)\vec{B} + \frac{1}{c}(\vec{\beta}\cdot\nabla)(\vec{E}\times\vec{S})\right\}\end{aligned} \tag{39}$$

Remembering that $\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$ the first and fifth terms inside the braces combine to give

$$\begin{aligned} \nabla(\vec{B}\cdot\vec{S}) - \frac{1}{c^2}\vec{S}\times\frac{\partial \vec{E}}{\partial t} &= (\vec{S}\cdot\nabla)\vec{B} + \vec{S}\times(\nabla\times\vec{B}) - \vec{S}\times(\nabla\times\vec{B}) \\ &= (\vec{S}\cdot\nabla)\vec{B}, \end{aligned} \tag{40}$$

$$\begin{aligned} \vec{F} = e(\vec{E}+\vec{v}\times\vec{B}) + \frac{ge}{2m}\Bigg\{ &(\vec{S}\cdot\nabla)\vec{B} + (\vec{\beta}\cdot\vec{S})\left[-\nabla(\vec{\beta}\cdot\vec{B}) + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} + (\vec{\beta}\cdot\nabla)\vec{B}\right] \\ &+ \frac{1}{c}\nabla[\vec{\beta}\cdot(\vec{S}\times\vec{E})] - \frac{1}{c}(\vec{\beta}\cdot\nabla)(\vec{S}\times\vec{E})\Bigg\} \end{aligned} \tag{41}$$

Using the identity for an arbitrary vector $\vec{\xi}$:

$$(\vec{\beta}\cdot\nabla)\vec{\xi} - \nabla(\vec{\beta}\cdot\vec{\xi}) = \vec{\beta}\times(\nabla\times\xi), \tag{42}$$

the force further simplifies to

$$\begin{aligned} \vec{F} &= e(\vec{E}+\vec{v}\times\vec{B}) \\ &+ \frac{ge}{2m}\left\{(\vec{S}\cdot\nabla)\vec{B} - \frac{1}{c}(\vec{S}\cdot\nabla)(\vec{\beta}\times\vec{E}) + \frac{1}{c}(\vec{\beta}\cdot\vec{S})\left(\frac{\partial \vec{B}}{\partial t} - \frac{1}{c}\vec{\beta}\times\frac{\partial \vec{E}}{\partial t}\right)\right\}. \end{aligned} \tag{43}$$

The first part of this force is the usual Lorentz force on a charged particle, whereas the second part, which is equivalent to Eq. (22), is due to the spin or magnetic moment of the particle.

Thoughts on the deficiencies of this treatment

The BMT equation* cannot be derived from the Lagrangian of Eq. (25) or the Hamiltonian Eq. (33), since they are actually incomplete. In order to write a Hamiltonian for the BMT equation, we need to have a rotational energy term simplistically something like

$$\frac{S^2}{2I}, \tag{44}$$

where $1/I$ represents the inverse of the moment of inertia tensor. For the Lagrangian the necessary term would simplistically look something like

$$\frac{1}{2}I\dot{\theta}^2, \tag{45}$$

where $\dot{\theta}$ represents the angular velocity of the particle. In special relativity, the angular velocity may be represented by the antisymmetric tensor, $\Omega^{\alpha\beta}$. I have not figured out how

* V. Bargmann, Louis Michel, and V. L. Telegdi, Phys. Rev. Lett., 435, 2 (1959).

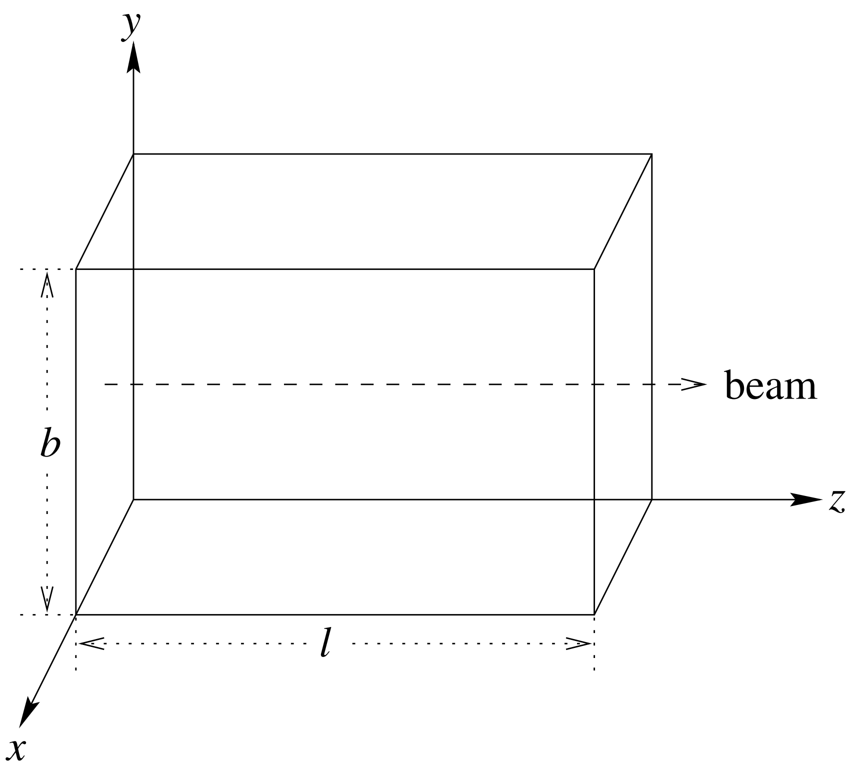

Figure. 1 A simple rectangular rf cavity. The beam moves parallel to the z-axis at a height of $y = b/2$.

to deal with the analog of the moment of inertia, although, it may need to be a tensor of rank four.

If the rotational energy term were included, then a constraint must be used to keep the magnitude of the angular momentum constant, i. e. $S_\alpha S^\alpha$ must be held constant. Without this constraint, we should expect that, in general, a magnetic moment can change magnitude. If this were not the case, then transformers would not work, since the secondary circuit may be considered to be a magnetic moment sitting in a changing EM field.

Example of a longitudinally polarized particle in a TE cavity

For a longitudinally polarized particle the relativistic equivalent of the Stern-Gerlach force is then

$$\vec{F}_{SG} = \gamma\mu^\diamond \left[\frac{\partial \vec{B}}{\partial z} + \frac{\beta}{c}\frac{\partial \vec{B}}{\partial t} - \frac{\beta}{c}\hat{z} \times \left(\frac{\beta}{c}\frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{E}}{\partial z} \right) \right], \tag{46}$$

where the direction of motion is parallel to the z-axis, and

$$\vec{\mu}^\diamond = \frac{ge}{mc}\vec{S}^\diamond, \tag{47}$$

for the magnetic moment of the particle in the rest system.

The vector potential for simple rectangular cavity with a TE_{0mn} mode may be written

$$\mathbf{A} = \begin{pmatrix} 0 \\ -B_0 \frac{b}{m\pi} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{l} \sin\omega t \\ 0 \\ 0 \end{pmatrix}, \tag{48}$$

where b is the cavity height (y-dimension) and l is the cavity length (See Fig. 1), and n and m are positive integers. The frequency of the TE_{0mn} mode is

$$f = \frac{\omega}{2\pi} = \frac{c}{2}\sqrt{\left(\frac{m}{b}\right)^2 + \left(\frac{n}{l}\right)^2}. \tag{49}$$

Since

$$\vec{B} = \nabla \times \vec{A}, \quad \text{and} \tag{50a}$$

$$\vec{B} = -\frac{\partial \vec{A}}{\partial t} - \nabla\phi, \tag{50b}$$

the electromagnetic field components in the lab are

$$E_x = \frac{b\omega}{m\pi} B_0 \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{l} \cos\omega t \tag{51a}$$

$$E_y = 0 \tag{51b}$$

$$E_z = 0 \tag{51c}$$

$$B_x = 0 \tag{52a}$$

$$B_y = -\frac{nb}{ml} B_0 \sin\frac{m\pi y}{b} \cos\frac{n\pi z}{l} \sin\omega t \tag{52b}$$

$$B_z = B_0 \cos\frac{m\pi y}{b} \sin\frac{n\pi z}{l} \sin\omega t \tag{52c}$$

Substituting into Eq. (46) gives

$$F_x = 0 \tag{52c}$$

$$\begin{aligned} F_y = \gamma\mu^\diamond B_0 \frac{b}{m\pi} \sin\frac{m\pi y}{b} & \left[\left(\frac{n^2\pi^2}{l^2} + \frac{\beta^2\omega^2}{c^2}\right) \sin\frac{n\pi z}{l} \sin\omega t \right. \\ & \left. - 2\frac{n\pi}{l}\frac{\beta\omega}{c} \cos\frac{n\pi z}{l} \cos\omega t\right] \end{aligned} \tag{52c}$$

$$F_z = \gamma\mu^\diamond B_0 \cos\frac{m\pi y}{b} \left[\frac{n\pi}{l} \cos\frac{n\pi z}{l} \sin\omega t + \frac{\beta\omega}{c} \sin\frac{n\pi z}{l} \cos\omega t\right] \tag{52c}$$

Note that the longitudinal force along the beam axis ($y = b/2$) is identically zero for odd m, so the only interesting modes are those with even $m > 0$.

For a particle with longitudinal magnetic moment moving along the z-axis with velocity $v = \beta c$, the longitudinal component of force (for even $m > 0$) is

$$\begin{aligned} F_z &= (-1)^{m/2}\gamma\mu^\diamond B_0 \left\{\frac{n\pi}{l} \cos\frac{n\pi z}{l} \sin\left(\phi_0 + \frac{\omega z}{\beta c}\right) + \frac{\beta\omega}{c} \sin\frac{n\pi z}{l} \cos\left(\phi_0 + \frac{\omega z}{\beta c}\right)\right\} \\ &= (-1)^{m/2}\frac{\gamma\mu^\diamond B_0}{2} \left\{\frac{n\pi}{l}\left[\sin\left(\frac{\omega l + n\pi\beta c}{\beta c l} z + \phi_0\right) + \sin\left(\frac{\omega l - n\pi\beta c}{\beta c l} z + \phi_0\right)\right]\right. \\ &\qquad \left. + \frac{\beta\omega}{c}\left[\sin\left(\frac{n\pi\beta c + \omega l}{\beta c l} z + \phi_0\right) + \sin\left(\frac{n\pi\beta c - \omega l}{\beta c l} z - \phi_0\right)\right]\right\}, \end{aligned} \tag{53}$$

where ϕ_0 is the phase of the rf voltage when the particle enters the cavity at $z = 0$. The energy increase to the particle from the cavity is then

$$
\begin{aligned}
\Delta U &= \int_0^l F_z \, dz \\
&= (-1)^{m/2} \frac{\gamma \mu^\diamond B_0}{2} \left\{ \frac{n\pi\beta c}{\omega l + n\pi\beta c} \left[\cos\phi_0 - \cos\left(\frac{\omega l + n\pi\beta c}{\beta c} + \phi_0 \right) \right] \right. \\
&\qquad + \frac{n\pi\beta c}{\omega l - n\pi\beta c} \left[\cos\phi_0 - \cos\left(\frac{\omega l - n\pi\beta c}{\beta c} + \phi_0 \right) \right] \\
&\qquad + \frac{\beta^2 \omega l}{n\pi\beta c + \omega l} \left[\cos\phi_0 - \cos\left(\frac{n\pi\beta c + \omega l}{\beta c} + \phi_0 \right) \right] \\
&\qquad \left. + \frac{\beta^2 \omega l}{n\pi\beta c - \omega l} \left[\cos\phi_0 - \cos\left(\frac{n\pi\beta c - \omega l}{\beta c} - \phi_0 \right) \right] \right\}. \qquad (54)
\end{aligned}
$$

In the above equation, the terms in brackets are all equivalent, i. e.,

$$
[\cdots] = \cos\phi_0 - (-1)^n \cos\left(\frac{\omega l}{\beta c} + \phi_0 \right), \qquad (55)
$$

so then energy integral simplifies to

$$
\begin{aligned}
\Delta U &= (-1)^{m/2} \frac{\gamma \mu^\diamond B_0}{2} \left[\frac{2\omega l n\pi\beta c}{(\omega l)^2 - (n\pi\beta c)^2} + \frac{2\beta^2 \omega l n\pi\beta c}{(n\pi\beta c)^2 - (\omega l)^2} \right] \\
&\qquad \left[\cos\phi_0 - (-1)^n \cos\left(\frac{\omega l}{\beta c} + \phi_0 \right) \right] \\
&= (-1)^{m/2} \frac{\mu^\diamond B_0}{\gamma} \frac{R}{1 - R^2} \left[\cos\phi_0 - (-1)^n \cos\left(\frac{n\pi}{R} + \phi_0 \right) \right], \qquad (56)
\end{aligned}
$$

where, with the help of Eq. (49),

$$
R = \frac{n\pi\beta c}{\omega l} = \frac{\beta}{\sqrt{1 + \left(\frac{ml}{nb}\right)^2}}. \qquad (57)
$$

For large γ this gives a γ dependence roughly like

$$
\frac{1}{\gamma} \frac{R}{1 - R^2} = \frac{\beta \sqrt{1 + \left(\frac{ml}{nb}\right)^2}}{\gamma \left(\frac{ml}{nb}\right)^2 + \gamma^{-1}} \propto \frac{1}{\gamma}, \qquad (58)
$$

since $m > 0$ and $n > 0$. The contribution from the cosine terms is just a factor between ± 2.

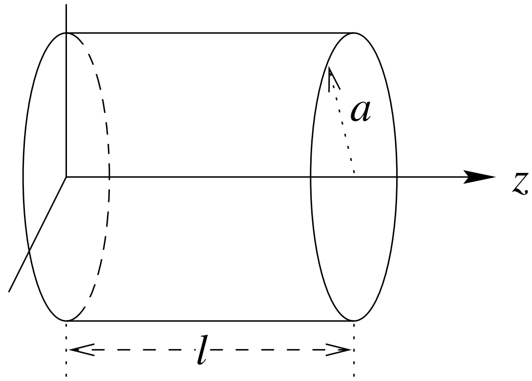

Figure. 2 A simple cylindrical cavity. The beam moves parallel to the z-axis with $r = 0$.

Consider now the simple cylindrical cavity† shown in Fig. 2. The TE_{0mn} mode has a longitudinal magnetic field of

$$B_z = B_0\, J_0\left(\frac{X'_{0m} r}{a}\right) \sin\frac{n\pi z}{l}\, \sin\omega t, \tag{59}$$

where X'_{0m} is the m^{th} root of the Bessel function derivative $J'_0(x)$, and m and n are positive integers. The resonant frequency for the TE_{0mn} mode is given by

$$\omega = c\sqrt{\left(\frac{X'_{0m}}{a}\right)^2 + \left(\frac{n\pi}{l}\right)^2}. \tag{60}$$

The longitudinal force along the axis is then

$$\begin{aligned} F_z &= \gamma\mu^\diamond B_0 \left\{\frac{n\pi}{l}\cos\frac{n\pi a}{l}\, \sin\omega t + \frac{\beta\omega}{c}\, \sin\frac{n\pi z}{l}\, \cos\omega t\right\} \\ &= \frac{\gamma\mu^\diamond B_0}{2}\left\{\frac{n\pi}{l}\left[\sin\left(\frac{\omega l + n\pi\beta c}{\beta c l} z + \phi_0\right) + \sin\left(\frac{\omega l - n\pi\beta c}{\beta c l} z + \phi_0\right)\right]\right. \\ &\qquad \left. + \frac{\beta\omega}{c}\left[\sin\left(\frac{n\pi\beta c + \omega l}{\beta c l} z + \phi_0\right) + \sin\left(\frac{n\pi\beta c - \omega l}{\beta c l} z - \phi_0\right)\right]\right\}, \end{aligned} \tag{61}$$

which is almost identical to Eq. (53) except for the initial factor of $(-1)^{m/2}$. Integrating to obtain the work done by the cavity field gives

$$\Delta U = (-1)^{m/2}\, \frac{\mu^\diamond B_0}{\gamma}\, \frac{R}{1 - R^2}\left[\cos\phi_0 - (-1)^n \cos\left(\frac{n\pi}{R} + \phi_0\right)\right], \tag{62}$$

with

$$R = \frac{n\pi\beta c}{\omega l} = \frac{\beta}{\sqrt{1 + \left(\frac{X'_{0m} l}{n\pi a}\right)^2}}. \tag{63}$$

† Samuel Y. Liao, *Microwave Devices and Circuits*, Prentice Hall (1985).

The γ-dependence is again similar to Eq. (58):

$$\frac{1}{\gamma}\,\frac{R}{1-R^2}=\frac{\beta\sqrt{1+\left(\frac{X'_{0m}l}{nb}\right)^2}}{\gamma\left(\frac{X'_{0m}l}{n\pi a}\right)^2+\gamma^{-1}}\propto\frac{1}{\gamma}, \tag{64}$$

since $X'_{0m}\neq 0$.

Revisions

1 Some typographical errors have been corrected in Eqs. (21a& b).
2 Changed $J_0(x)$ to $J'_0(x)$ in first line after Eq. (59).
3 Replaced $\vec{x}$ by ξ just before Eq. (42).
4 Replaced "covariant coordinates" by "contravariant contravariant" on first page to match the standard definition.