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Longitudinal phase space parameters

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1 Introduction

Occasionally there is minor confusion about how longitudinal phase space parameters, such as bunch area S, emittance ϵ_s, beta function β_s, and tune Q_s, are related. Although it rarely takes much time to clear up each case of confusion, the exercise can be frustrating, and the integrated amount of time that is wasted continues to accumulate. This, then, is our attempt to unambiguously state those relations, in a single convenient reference location.

2 Parametric relationships

2.1 Bunch area

The canonical coordinates often used to study longitudinal motion in an RF bucket are (ϕ_{RF}, W), where ϕ_{RF} is the RF phase of a test particle or ion, and

$$W = \frac{E - E_s}{\omega_{RF}} \tag{1}$$

is the total energy offset of the test particle relative to the synchronous particle, scaled by ω_{RF}, the angular frequency of the RF system [1, 2]. The bunch area is defined in this coordinate frame. If the bunch motion is linear - nowhere near the bucket separatrix - then the **RMS bunch area** is conveniently defined as

$$S_{RMS} = \frac{\pi}{A}\,\sigma_W \sigma_\phi \qquad [\text{eV s/u}] \tag{2}$$

where σ_W and σ_ϕ are the standard deviations of the distribution, and the mass number, A, is an *integer*. For protons, $A = 1$ and the units are [eV s], not [eV s/u]. If the distribution is bi-gaussian, the **95% bunch area** is given by

$$S_{95} = \frac{6\pi}{A}\,\sigma_W \sigma_\phi \qquad [\text{eV s/u}] \tag{3}$$

Note that the factor of π is usually *explicitly* included in numerical values of the bunch area. For example, it is common to see "$S_{RMS} = 0.1$ [eV s]", and rare to see "$S_{RMS} = 0.032\,\pi$ [eV s]", expressions which are (approximately) identical.

2.2 Emittance

The RMS bunch length σ_s and the RMS relative momentum spread σ_p are very often the practical parameters of choice, rather than σ_W and σ_ϕ. In direct analogy to transverse phase space, the **normalized longitudinal emittance** ϵ_s and the **longitudinal beta function** β_s are defined such that

$$\sigma_s = \sqrt{\frac{\beta_s \epsilon_s}{\beta\gamma}} \tag{4}$$

$$\sigma_p = \sqrt{\frac{\epsilon_s}{\beta_s(\beta\gamma)}} \tag{5}$$

where $\beta\gamma$ is the usual Lorentz factor. These equations may also be written

$$\epsilon_s = \sigma_s \sigma_p (\beta\gamma) \tag{6}$$

$$\beta_s = \frac{\sigma_s}{\sigma_p} \tag{7}$$

It is readily shown that

$$\sigma_W \sigma_\phi = \frac{m_0 c^2}{c} (\beta\gamma)\ \sigma_s \sigma_p \tag{8}$$

where c is the speed of light, and m_0 is the rest mass per nucleon - the total rest mass divided by A. Table 1 in the last section of this paper records m_0 values for prominent ion species.

The RMS emittance is therefore related to the 95% area by

$$\epsilon_s = \frac{1}{6\pi} \frac{c}{m_0 c^2} S_{95} \tag{9}$$

If the test particle is a proton, then

$$\epsilon_s\ [\mathrm{m}] = 0.01695 \times S_{95}\quad [\mathrm{eV\ s}] \tag{10}$$

or, equivalently, the RMS emittance is related to the RMS area by

$$\epsilon_s\ [\mathrm{m}] = 0.1017 \times S_{RMS}\ [\mathrm{eV\ s}] \tag{11}$$

Strictly speaking, the coefficients in the last two equations above should be modified for ion species other than protons. In practice, the rest mass values listed in Table 1 show that, if the same coefficients are used verbatim for all ion species, then the maximum error incurred is only about 2%. For most practical purposes this is negligible.

2.3 Beta function

The only parameter left hanging at this point is the longitudinal beta function, which can be shown [3] to be given by the remarkably simple expression

$$\beta_s = \frac{C}{2\pi} \frac{|\eta|}{Q_s} \tag{12}$$

where C is the circumference of the machine, and Q_s is the synchrotron tune. The slip factor (or momentum compaction factor) η is given by

$$\eta = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2} \tag{13}$$

where γ_T is the transition gamma of the machine. For RHIC, with $C = 3833.845$ [m] and $\gamma_T = 22.89$, if it is assumed that $\gamma \gg \gamma_T$, then

$$\beta_{s,RHIC} \approx \frac{1.165}{Q_s} \quad [m] \tag{14}$$

Note that the synchrotron tune is of order $Q_s \sim 10^{-3}$. If the slip factor is considered to be a fixed property of the transverse optics, then the sole independent variable is Q_s. The synchrotron tune is (arguably) a much more natural independent parameter than the RF voltage V_{RF}, or the voltage slope V'_{RF}.

2.4 RF voltage

With t the turn number, $\delta = \Delta p/p$ the off momentum parameter, and s the longitudinal displacement, small amplitude motion in a stationary bucket is described by

$$\frac{ds}{dt} = -\eta C \quad \delta \tag{15}$$

$$\frac{d\delta}{dt} = \frac{1}{\beta^2} \frac{Z}{A} \frac{eV'_{RF}}{E_n} \quad s \tag{16}$$

where Z is the atomic number and E_n is the nominal *total* energy per nucleon. After solving for the synchrotron tune in the relativistic limit when $\beta \approx 1$, the required value of the voltage slope is given by

$$V'_{RF} = \frac{(2\pi Q_s)^2}{\eta C} \frac{A}{Z} \frac{E_n}{e} \tag{17}$$

The required RF voltage V_{RF} is related to the voltage slope through

$$|V'_{RF}| = \frac{2\pi}{\lambda_{RF}} V_{RF} \tag{18}$$

where λ_{RF} is the RF wavelength. Putting all this together gives, finally

$$V_{RF} = \lambda_{RF} \frac{2\pi Q_s^2}{|\eta| C} \frac{A}{Z} \frac{E_n}{e} \tag{19}$$

3 Rest mass per nucleon

Except for Carbon ions, the rest mass per nucleon m_0 is NOT the same as m_u, the atomic mass unit. One atomic mass unit is defined to be 1/12 of the mass of the most abundant isotope of Carbon. The value for m_u recorded in Table 1 was published by NIST in 1986. Note that the mass number, A, is the number of nucleons in an ion - an integer.

Name	Symbol	Atomic number Z	Mass number A	Rest mass per nucleon m_0 [GeV/c^2]
Proton	p	1	1	.93827
Deuteron	d	1	2	.93781
Carbon	C	6	12	.93149432
Oxygen	O	8	16	.93093
Silicon	Si	14	28	.93046
Copper	Cu	29	63	.92022
Iodine	I	53	127	.93058
Gold	Au	79	197	.93113

Table 1: The rest mass per nucleon, for various ion species.

References

[1] See, for example, Conte & Mackay, "An Introduction to the Physics of Particle Accelerators", p. 116 et seq.

[2] Jie Wei, "Longitudinal Dynamics of the Non-Adiabatic Regime of Alternating Gradient Synchrotrons", PhD Thesis, Stony Brook, 1990.

[3] See, for example, Conte & Mackay (ibid), eqns 7.32, 7.103, 7.106 and 7.107, on pages 113, 125, and 126.