Mathcad file for estimates of electron cooling of RHIC beams and its effects on the luminosity of electron - nucleaon and (to be added) nucleon-nucleon collisions in RHIC. The file has been written by Vitaly Yakimenko and Ilan Ben-Zvi, BNL, with contributions from Vasily Parkhomchuk, BINP
and Alexey Burov, FNAL.
This is work in progress. Current version September 6, 2000
Define:

$Gold := \text{"Gold"}$

$Proton := \text{"Proton"}$

Choose ion for the calculation:

$Ion := Gold$

Speed of light [m/s]:

$c := 2.998 \cdot 10^{8}$

Electron charge [Coulomb]:

$q := 1.6022 \cdot 10^{-19}$

$mc2 := 0.511 \cdot 10^{6}$

Electron rest mass [eV]

$Mc2 := 938.28 \cdot 10^{6}$

Proton rest mass [eV]

Classical radius of electron [m]:

$re := 2.818 \cdot 10^{-15}$

Classical radius of proton [m]:

$rp := re \cdot \frac{mc2}{Mc2}$

Bolzmann's constant [eV/K]:

$k := \frac{1.381 \cdot 10^{-23}}{1.602 \cdot 10^{-19}}$

Velocity /c as a function of g:

$\beta(\gamma) := \sqrt{1 - \gamma^{-2}}$

$\varepsilon_0 := 8.842 \times 10^{-12}$

Atomic number of the ion

$A := \begin{cases} 1 & \text{if } Ion = Proton \\ 197 & \text{if } Ion = Gold \end{cases}$

$A = 197$

Charge of the ion

$Z := \begin{cases} 1 & \text{if } Ion = Proton \\ 79 & \text{if } Ion = Gold \end{cases}$

$Z = 79$

Ion energy per nucleon [GeV/AMU]

$E_{nuc} := \begin{cases} 250 & \text{if } Ion = Proton \\ 100 & \text{if } Ion = Gold \end{cases}$

Ion beam energy [GeV]:

$Ei := E_{nuc} \cdot A$

$\gamma i := \frac{E_{nuc}}{0.938}$

$\gamma i = 106.61$

Electron beam energy [GeV]:

$Ee := 10$

$\gamma e := \frac{Ee}{0.000511}$

$\gamma e = 1.957 \times 10^{4}$

Number of Ions per bunch:

$Ni := \begin{cases} 2 \cdot 10^{11} & \text{if Ion = Proton} \\ 2 \cdot 10^{9} & \text{if Ion = Gold} \end{cases}$

Number of Electrons per bunch:

$Ne := 3 \cdot 10^{10}$

$Q := q \cdot Ne$

$Q \cdot 10^{9} = 4.807$

Ions emittance, rms, [m*rad]

$\varepsilon i := \begin{cases} 3.25 \cdot 10^{-9} & \text{if Ion = Proton} \\ 2.4 \cdot 10^{-8} & \text{if Ion = Gold} \end{cases}$

$\varepsilon in := \varepsilon i \cdot \gamma i$

$\varepsilon in = 2.559 \times 10^{-6}$

Electron's emittance [m*rad]

$\varepsilon e := 3 \cdot 10^{-9}$

$\varepsilon en := \varepsilon e \cdot \gamma e$

$\varepsilon en = 5.871 \times 10^{-5}$

Ion beam energy spread

$\delta i := 4 \cdot 10^{-4}$

Ion beam length [m]

$\sigma i := \begin{cases} 0.1 & \text{if Ion = Proton} \\ 0.2 & \text{if Ion = Gold} \end{cases}$

Electron Beam length [m]

$\sigma e := 0.1$

Bunch spacing [m]

$l := \frac{3834}{360}$

$l = 10.65$

Single bunch area (longitudinal, in eV s/A):

$\frac{\sigma i}{\beta(\gamma i) \cdot c} \cdot 10^{9} \cdot E_{nuc} \cdot \delta i = 0.027$

b - functions of ions at interaction point [m]:

$$\beta i := \begin{cases} 0.36 \cdot \dfrac{\varepsilon in}{10^{-6}} & \text{if } Ion = Proton \\ 0.36 \cdot \dfrac{\varepsilon in}{10^{-6}} & \text{if } Ion = Gold \end{cases}$$

The idea is to keep sigma prime star constant

Tunes :

$\nu x := 28.18$

$\nu y := 29.18$

Circumserence [m]:

$C := 3834$

Dispersion in m

$Dx := 0.600$

Set beam sizes to be equal:

$$\beta e := \frac{\beta i \cdot \varepsilon i}{\varepsilon e}$$

$\beta e = 7.369$

RMS beam size of electrons and ions at IP

$\sqrt{\varepsilon i \cdot \beta i} = 1.487 \times 10^{-4}$

Cooler parameters:

Beam charge[Coulomb]:

$$Qc := \begin{cases} 10 \cdot 10^{-9} & \text{if } Ion = Proton \\ 10 \cdot 10^{-9} & \text{if } Ion = Gold \end{cases}$$

Cooling length [m]

$l_c := 30$

$B := 1.$

Cooler field (Tesla)

$\beta ic := 60$

beta-function of ion beam in the cooling region [m]:

$$R_ := \frac{C}{2 \cdot \pi}$$

Averages:

$R_ = 610.2$

$$\beta ih_ := \frac{R_}{\nu x}$$

$\beta ih_ = 21.654$

$$\beta iv_ := \frac{R_}{\nu y}$$

$\beta iv_ = 20.912$

$$\eta i_ := \frac{R_}{\nu x^2}$$

$\eta i_ = 0.768$

Average horizontal betatron size

$\sigma ih := \sqrt{\varepsilon i \cdot \beta ih_}$

$\sigma ih = 7.209\times 10^{-4}$

$\sigma iv := \sqrt{\varepsilon i\cdot \beta iv_}$

$\sigma iv = 7.084\times 10^{-4}$

Average vertical betatron size (m)

Hour-glass reduction factor:

$$F(\zeta) := \frac{2}{\sqrt{\pi}}\cdot\int_0^{10}\frac{\exp\left(-s^2\right)}{1+\zeta^2\cdot s^2}\,ds$$

$$F\left(\frac{\sigma i}{\beta i}\right) = 0.978$$

$$F\left(\frac{\sigma e}{\beta e}\right) = 1$$

Ion = "Gold"

Ion - ion luminosity [cm**-2 sec**-2]:

$$LI := \frac{c}{l}\cdot\frac{Ni^2\cdot 10^{-4}}{4\cdot\pi\cdot\beta i\cdot\varepsilon i}\cdot F\left(\frac{\sigma e}{\beta e}\right)$$

$LI = 4.053\times 10^{28}$

Electron - ion luminosity (assuming same IP beam size for electrons and ions):

Short bunch Luminosity[cm^-2 s^-1]:

$$L0 := \frac{c}{l}\cdot\frac{Ni\cdot Ne\cdot 10^{-4}}{4\cdot\pi\cdot\beta i\cdot\varepsilon i}\cdot F\left(\frac{\sigma e}{\beta e}\right)$$

$L0 = 6.079\times 10^{29}$

Electron - ion Beam-Beam tune shift

$$\xi e := \frac{Ni\cdot Z\cdot re}{4\cdot\pi\cdot\varepsilon e}\cdot\frac{1}{\gamma e}$$

$\xi e = 0.604$

Ion - electron Beam-Beam tune shift:

$$\xi i := \frac{Ne\cdot Z\cdot rp}{4\cdot\pi\cdot A\cdot\varepsilon i}\cdot\frac{1}{\gamma i}$$

$\xi i = 5.742\times 10^{-4}$

Ion - ion Beam-Beam tune shift:

$$\xi ii := \frac{Ni\cdot Z^2\cdot rp}{4\cdot\pi\cdot A\cdot\varepsilon i}\cdot\frac{1}{\gamma i}$$

$\xi ii = 3.024\times 10^{-3}$

Lasslett tune shift

$$\Delta\nu L := \frac{Z^2}{A}\cdot\frac{Ni\cdot rp}{4\cdot\pi\cdot\gamma i^3\cdot\varepsilon i}\cdot\frac{C}{\sigma i\cdot\sqrt{2\cdot\pi}}$$

$\Delta\nu L = 2.035\times 10^{-3}$

Disruption parameter of electron beam by ions:

$$D_e := 2\cdot\xi e\cdot\frac{\sigma i}{\beta i}$$

$$D_e = 0.262$$

Angular spread generated by disruption:

$$xprime := \frac{D_e\cdot\sqrt{\beta i\cdot\varepsilon i}}{\sigma i}$$

$$xprime = 1.948\times 10^{-4}$$

$$\sigma pe := \sqrt{\frac{\varepsilon i}{\beta i}}$$

$$\sigma pe = 1.614\times 10^{-4}$$

Electron beam's intrinsic angular spread:

Ratio of disruption and e-beam intrinsic angular spreads:

$$DR := \frac{xprime}{\sigma pe}$$

$$DR = 1.207$$

Multiple IBS [1/s] (based on expression developed by Jie Wei, taken from a manuscript by W. Fischer et.al,:

$$Libs := 10$$

$$I(x) := \begin{cases} \frac{1}{\sqrt{x\cdot(x-1)}}\cdot atanh\left(\sqrt{\frac{x-1}{x}}\right) & \text{if } (x \ge 1) \\ \left[\frac{1}{\sqrt{x\cdot(1-x)}}\cdot atan\left(\sqrt{\frac{1-x}{x}}\right)\right] & \text{otherwise} \end{cases}$$

$$FF(x) := \frac{(1+2\cdot x)\cdot I(x) - 3}{1-x}$$

$$Factor := \frac{Z^4\cdot Ni}{A^2}\cdot\frac{rp^2\cdot Libs\cdot c}{8\cdot\pi\cdot\gamma i^4\,\varepsilon i^2\cdot\sigma i\cdot\delta i}$$

$$d := \frac{Dx\cdot\delta i}{\sqrt{\sigma ih^2 + Dx^2\cdot\delta i^2}}$$

$$a := \frac{\beta ih_\cdot d}{Dx\cdot\gamma i}$$

$$b := \frac{\beta iv_\cdot\sigma ih}{\beta ih_\cdot\sigma iv}\cdot a$$

$$d = 0.316$$

$$a = 0.107$$

$$b = 0.105$$

$$\chi := \frac{a^2 + b^2}{2}$$

$$\chi = 0.011$$

$$\Lambda h := Factor\cdot FF(\chi)\cdot\left(d^2 - \frac{a^2}{2}\right)$$

$$\Lambda h^{-1} = 9.901\times 10^3$$

$\Lambda v := -\text{Factor} \cdot FF(\chi) \cdot \frac{b^2}{2}$

$\Lambda v^{-1} = -1.687 \times 10^5$

$\Lambda l := \text{Factor} \cdot FF(\chi) \cdot (1 - d^2)$

$\Lambda l^{-1} = 1.034 \times 10^3$

Electron cooling beam parameters:

$Ec := \frac{mc2 \cdot \gamma i}{10^6}$

$Ec = 54.478$

Energy [MeV]

Number of electrons:

$Nc := \frac{Qc}{q}$

$Nc = 6.241 \times 10^{10}$

Beam length [m]:

$\sigma c := \sigma i$

$\sigma i = 0.2$

Cooler peak current (Amperes):

$Ic := Qc \cdot \frac{c}{\sigma c \cdot \sqrt{2\pi}}$

$Ic = 5.98$

Avg. Cooler current (Amperes):

$Ica := Qc \cdot \frac{c}{l}$

$Ica = 0.282$

$\delta c := 1 \cdot 10^{-3} \cdot \frac{0.003}{\sigma c}$

Energy spread [DE/E]:

$\delta c = 1.5 \times 10^{-5}$

Normalized emittance [m*rad]:

$\varepsilon cn := \frac{Qc}{1.6 \cdot 10^{-9}} \cdot 6 \cdot 10^{-6}$

$\varepsilon cn = 3.75 \times 10^{-5}$

b-function of ion beam in the cooling region [m]:

$\beta ic := 60$

b-function of the cooling beam [m]:

$\beta c := \frac{\beta ic \cdot \varepsilon i \cdot \gamma i}{\varepsilon cn}$

$\beta c = 4.094$

Cooling beam radius [m]

$rc := \sqrt{\frac{\varepsilon cn \cdot \beta c}{\gamma i \cdot \beta(\gamma i)}}$

$rc = 1.2 \times 10^{-3}$

$$nc := \frac{Nc}{\pi \cdot rc^2 \cdot \sigma c}$$

$nc = 6.898 \times 10^{16}$

Beam density [1/m^3] (in lab frame):

Plasma frequency (in moving frame)

$$\omega_p := c \cdot \sqrt{4 \cdot \pi \cdot \frac{nc}{\gamma i} \cdot re}$$

Compare Debye shielding time to time of flight thru the cooler section (both in moving frame):

$$\frac{1}{\omega_p} = 6.968 \times 10^{-10}$$

$$\frac{l_c}{c \cdot \gamma i} = 9.386 \times 10^{-10}$$

Elecron cooling region length ratio:

$$\eta := \frac{l_c}{C}$$

$\eta = 7.825 \times 10^{-3}$

Electron longitudinal temperature [eV]:

$$Tcl := mc2 \cdot \delta c^2$$

$Tcl = 1.15 \times 10^{-4}$

Electron transverse temperature [eV]:

$$Tct := \frac{mc2 \cdot \varepsilon cn \cdot \gamma i}{\beta c}$$

$Tct = 499.023$

Debye length:

$$L_D := \sqrt{\frac{\varepsilon_0}{q} \cdot \frac{k \cdot Tct \cdot \gamma i}{nc}}$$

$L_D = 6.057 \times 10^{-5}$

Ion beam longiudinal temp [eV]:

$$Til := Mc2 \cdot \beta(\gamma i)^2 \cdot \delta i^2$$

$Til = 150.112$

Ion beam vertical temp [eV]:

$$Tit := \frac{A \cdot Mc2 \cdot \beta(\gamma i)^2 \cdot \gamma i \cdot \varepsilon in}{\beta ic}$$

$Tit = 8.403 \times 10^5$

RMS speed [m/sec]:

$$Vrms(T, massc2) := \sqrt{\frac{T \cdot c^2}{massc2}}$$

$Vit := Vrms(Tit, Mc2)$

$Vit = 8.972 \times 10^6$

$Vct := Vrms(Tct, mc2)$

$Vct = 9.369 \times 10^{6}$

$Vil := Vrms(Til, Mc2)$

$Vil = 1.199 \times 10^{5}$

$Vcl := Vrms(Tcl, mc2)$

$Vcl = 4.497 \times 10^{3}$

Recombination:

The effective electron temperature (eV) consist from real temperature electron beam and additional part by motion of ions

Transversal electron temperature[eV]:

$Tct = 499.023$

Temperature due to ion speed[eV]:

$$Tic := \frac{0.511 \cdot 10^{6}}{2} \cdot \left(\frac{Vit}{c}\right)^{2}$$

$Tic = 228.809$

Speed due to magnetic field:

Regidity [T*m]:

$$H\rho := \frac{Ec}{300}$$

$H\rho = 0.182$

Magnet field [T]:

$B = 1$

Speed [m/sec]:

$$Vm := \frac{B \cdot c}{H\rho} \cdot rc$$

$Vm = 1.981 \times 10^{6}$

Temperature due to ion speed:

$$Tm := \frac{0.511 \cdot 10^{6}}{2} \cdot \left(\frac{Vm}{c}\right)^{2}$$

$Tm = 11.158$

$Teff := Tct + Tic + Tm$

$Teff = 738.99$

Recombination formula following Bell and Bell:

$$\alpha rec := 3.02 \cdot 10^{-19} \cdot \frac{Z^{2}}{\sqrt{Teff}} \cdot \left[\ln\left(\frac{11.32 \cdot Z^{2}}{\sqrt{Teff}}\right) + 0.14 \cdot \left(\frac{Teff}{Z^{2}}\right)^{\frac{1}{3}} \right]$$

The life time by recombination

$$\tau rec := \frac{\gamma i^{2}}{nc \cdot \alpha rec \cdot \eta}$$

$$\frac{\tau rec}{3600} = 10.637$$

Cooling time (s)

$$\Lambda_C := \ln\left(\frac{rc}{re} \cdot \beta(\gamma i) \cdot \gamma i \cdot \frac{\varepsilon in}{\beta ic}\right)$$

$\Lambda_C = 14.476$

$$\tau cool := \frac{\gamma i^2 \cdot A}{4 \cdot \pi \cdot rp \cdot re \cdot Z^2 \cdot nc \cdot \eta \cdot c \cdot \Lambda_C} \cdot \left(\frac{\varepsilon in \cdot \gamma i \cdot \beta(\gamma i)}{\beta ic} \right)^{\frac{3}{2}}$$

$$\tau cool = 27.315$$

Adaptation of Parkhomchuk's expression for cooling speed:

Velocity of the electron's thermal transverse motion

$$Vet := c \cdot \sqrt{\frac{Tct}{0.511 \cdot 10^6}}$$

Larmor radius

$$\rho Lar := \frac{0.511 \cdot 10^6}{c \cdot B} \cdot \frac{Vet}{c}$$

$$\rho Lar = 5.326 \times 10^{-5}$$

Minimum impact parameter

$$\rho min := \frac{re \cdot Z}{\left(\frac{Vit}{c} \right)^2}$$

$$\rho min = 2.486 \times 10^{-10}$$

Maximum impact parameter:

Ion beam size

$$\rho i := \sqrt{\varepsilon i \cdot \beta ic}$$

$$\rho i = 1.2 \times 10^{-3}$$

$$\rho ft := Vit \cdot \frac{l_c}{c}$$

$$\rho ft = 0.898$$

If limited by flight time:

$$\rho_D := \frac{Vit}{\omega_p}$$

If limited by Debye length:

$$\rho_D = 6.252 \times 10^{-3}$$

$$\rho max := \min(\rho i, \rho ft, \rho_D)$$

$$\Lambda_P := \ln\left(\frac{\rho max + \rho Lar + \rho min}{\rho Lar + \rho min} \right)$$

$$\Lambda_P = 3.158$$

$$\tau_P := \frac{\gamma i^2 \cdot A}{4 \cdot \pi \cdot rp \cdot re \cdot Z^2 \cdot nc \cdot \eta \cdot c \cdot \Lambda_P} \cdot \left(\frac{\varepsilon in \cdot \gamma i \cdot \beta(\gamma i)}{\beta ic} \right)^{\frac{3}{2}}$$

$$\tau_P = 125.202$$

Larmor period:

$$L_L := \gamma i \cdot 2 \cdot \pi \cdot .511 \cdot \frac{10^6}{B \cdot c}$$

$$L_L = 1.142$$

The maximum tune shift by beam beam with electron cooling

Exist a lots variant of fitting for maximal tune shift vs. the number of turns at the cooling time Ncool or at the life time of beam. Let used simplest variant popular at BINP (close to A.N. Skrinsky estimation) :

$$Ncool := \frac{c \cdot \beta(\gamma i)}{C} \cdot \tau cool$$

$$Ncool = 2.136 \times 10^{6}$$

$$\xi max := \frac{2 \cdot \pi}{Ncool^{\frac{1}{3}}}$$

$$\xi max = 0.049$$

Ion*Ion

$Ion = \text{"Gold"}$

Atomic number of the ion

$A = 197$

Charge of the ion

$Z = 79$

Ion energy per nucleon [GeV/AMU]

$E_{nuc} = 100$

Ion beam energy [GeV]:

$Ei = 1.97 \times 10^{4}$

Number of Ions per bunch:

$Ni = 2 \times 10^{9}$

Ions emittance, rms, [m*rad]

$\varepsilon in = 2.559 \times 10^{-6}$

(Nominal RHIC emittance, normalized rms value, is 2.3 x 10^{-6} m-rad for gold. Steve uses 1x10^-6)

Ion beam energy spread

$\delta i = 4 \times 10^{-4}$

Ion beam length [m]

$\sigma i = 0.2$

Bunch spacing [m]

$l = 10.65$

Ion - ion luminosity [cm**-2 sec**-2]:

$LI = 4.053 \times 10^{28}$

Ion - ion Beam-Beam tune shift:

$\xi ii = 3.024 \times 10^{-3}$

(The nominal RHIC luminosity for gold at 10^{9} particles/bunch is 9.6 x 10^{27})

Lasslett tune shift

$\Delta \nu L = 2.035 \times 10^{-3}$

$\Lambda h^{-1} = 9.901 \times 10^{3}$

Multiple IBS [s]

$\Lambda l^{-1} = 1.034 \times 10^{3}$

Peak Cooler current [A]:

$Ic = 5.98$

Avg. Cooler current [A]:

$Ica = 0.282$

Cooling time [s]

$\tau cool = 27.315$

Maximal tune shift with cooling

$\xi max = 0.049$

(Based on Skrinsky's formula).

The life time by recombination [hours]

$\frac{\tau rec}{3600} = 10.637$

Electron*Ion

$Ion = \text{"Gold"}$

Electron beam energy [GeV]:

$Ee = 10$

$\gamma e = 1.957 \times 10^{4}$

Number of Electrons per bunch:

$Ne = 3 \times 10^{10}$

$Q \cdot 10^{9} = 4.807$

Electron's emittance [m*rad]

$\varepsilon en = 5.871 \times 10^{-5}$

$\frac{\varepsilon en}{\gamma e} = 3 \times 10^{-9}$

Electron Beam length [m]

$\sigma e = 0.1$

Ions emittance, rms, [m*rad]

$\varepsilon in = 2.559 \times 10^{-6}$

$\varepsilon in \cdot 18 = 4.606 \times 10^{-5}$

Beta* of ions

$\beta i = 0.921$

Beta* of electrons

$\beta e = 7.369$

Electron - ion luminosity[cm^-2 s^-1]:

$L0 = 6.079 \times 10^{29}$

Electron - ion Beam-Beam tune shift

$\xi e = 0.604$

Electron disruption parameter

$D_e = 0.262$

Ion - electron Beam-Beam tune shift:

$\xi i = 5.742 \times 10^{-4}$

Electron beam current (amperes):

$\frac{Qc}{l} \cdot c = 0.282$

Ratio of disruption-induced and e-beam-intrinsic angular-spreads:

$DR = 1.207$