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Spin Note

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Angles from Spin Matrices

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ANGLES FROM SPIN MATRICES *

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1. Introduction

In spin tracking [1] we use 3×3 matrices. The spin is represented by a three dimensional real unitary vector. Spin matrices can be build from a theoretical expression [2] or from a magnetic field map of the magnetic device in consideration. For a helical snake, an expression for the matrix derived from a synthetic field [3] to zeroth order was given in Ref. [4].

We want to review briefly how a spin rotation matrix is constructed, to find handy expressions for the spin precession and spin flip angle and the orientation of the spin precession axis, to be used in tracking.

2. Rotation in Ordinary Space

A vector $\vec{S}$ rotates (precesses) around a fixed direction $\vec{b}$ in the ordinary cartesian space. The instantaneous variation of the vector $d\vec{S}$ is perpendicular to $\vec{b}$. $\vec{S}$ forms with $\vec{b}$ a constant angle. We write this condition as

$$\frac{d\vec{S}}{d\psi} = \vec{S} \times \vec{b} \tag{1}$$

where ψ is some angle. By performing the vector product in Eq. (1), the equation can be written explicitly as

$$\frac{d\vec{S}}{d\psi} = \hat{i}(S_1 b_3 - S_3 b_2) + \hat{j}(S_3 b_1 - S_1 b_3) + \hat{k}(S_1 b_2 - S_2 b_1) \tag{2}$$

with $\hat{i}, \hat{j}, \hat{k}$ three unitary vectors directed along the cartesian axes. b_1, b_2, b_3 and S_1, S_2, S_3 are the components of the two vectors along the axes, respectively.

Performing a second derivative on Eq. (1), using the equation again and doing a little algebra, obtain

$$\vec{S}'' = -\omega^2 \vec{S} + \vec{b}(\vec{b} \cdot \vec{S}) \tag{3}$$

with the definition

$$\omega^2 = b_1^2 + b_2^2 + b_3^2 \tag{4}$$

The dot product in Eq. (3) is a constant -indeed it gives the angle between $\vec{S}$ and $\vec{b}$-. This can be explictly verified by a non believer by performing the derivative and

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substitute the expression for $\vec{S'}$ from Eq. (2). Let's see:

$$\begin{aligned} K &= \vec{b}\cdot\vec{S} \\ K' &= b_1S_1' + b_2S_2' + b_3S_3' \\ &= b_1(S_1b_3 - S_3b_2) + b_2(s_3b_1 - S_1b_3) + b_3(S_1b_2 - S_2b_1) = 0 \\ K &= const. \end{aligned}$$

Write again Eq. (3) as follows

$$\vec{S''} + \omega^2\vec{S} = \vec{P} \tag{5}$$

It is a vector equation, equivalent to three scalar equations, all formally identical. Only the constant term on the r.h,s. is different from equation to equation. This term is $\vec{P} = K\vec{b}$. The general integral of Eq. (5) can be written as follows

$$\vec{S} = \vec{P} + \vec{Q}\sin\varphi + \vec{R}\cos\varphi \tag{6}$$

with the precession angle $\varphi = \omega\Delta\psi$

The three constant vectors $\vec{P}, \vec{Q}, \vec{R}$ can be found from the initial conditions. This done, the general solution can be written

$$\vec{S} = \vec{S_0} + \vec{S''}_0\frac{1-\cos\varphi}{\omega^2} + \vec{S'}_0\frac{\sin\varphi}{\omega} \tag{7}$$

Let us now write the three components of the solution, and make it in a matrix form. The matrix connects the components of the vector before and after the rotation. The matrix R is given in Ref. [2]. It can be written as the sum of a unitary matrix I, a symmetric matrix W and an anti symmetric matrix A as follows

$$R = I\ \cos\varphi + W\ \frac{1-\cos\varphi}{\omega^2} + A\ \frac{\sin\varphi}{\omega} \tag{8}$$

The individual matrices are

$$W = \begin{pmatrix} b_2^2 & b_1b_2 & b_1b_3 \\ b_2b_1 & b_2^2 & b_2b_3 \\ b_3b^1 & b_3b_2 & b_3^2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix} \tag{9}$$

3. Case of Spin Rotation

The vector $\vec{S}$ is now a spin vector and the environment is an accelerator. Make the usual convention for accelerator coordinates and take as axis No. 1 the radial axis (x), as axis No. 2 the vertical axis (y), and as axis No. 3 the longitudinal axis (z). The Spin precesses around some effective magnetic field in each element of the machine. The theory [2] based on the BMT equation for the spin, shows how the components of the vector $\vec{b}$ are related to the field and to the energy of the particle.

An important angle is the spin flipping angle μ, i.e. the angle formed between the final $\vec{S_1}$ and initial $\vec{S_0}$ orientation of the spin. The absolute value of this angle can be calculated from the scalar product

$$\cos\mu = \vec{S_1} \cdot \vec{S_0} \tag{10}$$

By definition the spin flipping angle cannot be larger than π.

Let us also define two angles for the orientation of $\vec{b}$ as follows. The latitude θ is the angle between the vector and the horizontal x, z plane. The azimuth ϕ is the angle formed by the projection of $\vec{b}$ on the horizontal plane with z. For instance, the axis of a Siberian snake for RHIC is located in the horizontal plane $\theta = 0$ and forms with z an angle of $\pm 45^0$

A spin rotation is defined by three angles, μ, θ, ϕ. We want to derive them from the elements of any spin matrix, e.g. a matrix calculated from a field map. Using the latitude and azimuth, the components of the precession axis are

$$\begin{aligned} b_1 &= \omega\cos\theta\sin\phi \\ b_2 &= \omega\sin\theta \\ b_3 &= \omega\cos\theta\cos\phi \end{aligned} \tag{11}$$

To write an explicit expression for the matrices of Eq. (8) let us use some compact notation

$$\begin{array}{lll} c = \cos\varphi & u = \cos\theta & p = \cos\phi \\ s = \sin\varphi \;, & v = \sin\theta \;, & q = \sin\phi \end{array} \tag{12}$$

and write

$$W = \begin{pmatrix} u^2q^2 & uvq & u^2pq \\ (.) & v^2 & uvp \\ (.) & (.) & u^2p^2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & up & -v \\ -(.) & 0 & uq \\ -(.) & -(.) & 0 \end{pmatrix} \tag{13}$$

Explicit expressions of the matrix elements are

$$\begin{array}{lll} R_{11} = c + u^2q^2(1-c) & R_{12} = u\left[vq(1-c)+p\right] & R_{13} = u^2pq(1-c) - vs \\ R_{21} = u\left[vq(1-c)-ps\right] & R_{22} = c + v^2(1-c) & R_{23} = u\left[vp(1-c)+qs\right] \\ R_{31} = u^2pq(1-c) + vs & R_{32} = u\left[vp(1-c)-qs\right] & R_{33} = c + u^2p^2(1-c) \end{array} \tag{14}$$

The angles can be calculated by combining the matrix elements in many different ways. Not all combinations are suitable for computers in all cases. In our case, with a precession axis on the horizontal plane, $v \approx 0$, and the total precession angle $\approx 180^0$, or $s \approx 0$, one should avoid all expressions that carry v or s in the denominator.

A suitable chain of calculation is

$$\begin{aligned} \tan 2\phi &= \frac{2W_{13}}{W_{33}-W_{11}} \\ \tan 2\theta &= -\frac{A_{12}A_{13}\sin\phi}{A_{12}A_{23}-A_{13}^2\sin\phi\cos\phi} \\ \cos\varphi &= \frac{A_{12}A_{23}}{W_{13}-1} \\ \cos\mu &= (R\vec{S_0})\cdot\vec{S_1} \end{aligned}$$

Note that the use of the arctan function[‡] in the calculation of the axis angles allows one to find the angles in the full circle $0^0 - 360^0$, while the precession angle φ will be only defined between 0^0 and 180^0. It is also useful to note that if the initial orientation of the spin is $(0,1,0)$ (vertical), we obtain

$$\cos\mu = W_{22}$$

4. Particular Cases

If the axis of precession lies in the horizontal plane, the matrices of Eq. (9) become

$$W = \frac{1}{2}\begin{pmatrix} 1-\cos 2\phi & 0 & \sin 2\phi \\ 0 & 0 & 0 \\ \sin 2\phi & 0 & 1+\cos 2\phi \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \cos\phi & 0 \\ -\cos\phi & 0 & \sin\phi \\ 0 & -\sin\phi & 0 \end{pmatrix} \tag{15}$$

In this case, the vertical (y) component of the spin $\vec{S}$ is not affected by the W or A matrices, but only by the term $I\cos\mu$ in Eq. (8).

If, moreover, the axis is at $\phi = 45^0$ or $= 180^0 + 45^0$, we have

$$W = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and if the axis is at $\phi = -45^0$ or $= 180^0 - 45^0$

$$W = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad A = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Finally, for a $\mu = 180^0$ flipping, with axis in the horizontal plane, at $\pm 45^0$, or $\mp 135^0$, respectively (RHIC snakes), it is

$$R = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & -1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}$$

5. References

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[‡] function atan2, in Fortran