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**Computation of the Harmonics in a Helically
Wound Multipole Magnet**

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MAGNET DIVISION NOTES

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Computation of the Harmonics in a Helically Wound Multipole Magnet
G.H. Morgan

The field of a beam transport magnet having spiral or helical windings has been examined by several authors [1,2,3]. The present treatment follows Caspi's [2] most closely, but uses the potential form of Ptitsin [3]. In regions where there is no current or iron, the field can be obtained from a scalar potential V which is assumed to be periodic in z , since the winding is also periodic in z with a pitch length L . Thus, $V = F(r) e^{in(\theta-kz)}$ in cylindrical coordinates, where $k = 2\pi/L$, and $F(r)$ is to be determined. V satisfies Laplace's equation, $\nabla^2 V = 0$. Substituting V into this, the differential equation for F is found to be a form of Bessel's equation:

$$r^2 \frac{\partial^2 F}{\partial r^2} + r \frac{\partial F}{\partial r} - (n^2 k^2 r^2 + n^2) F = 0.$$

This has the solution [4]

$$F_n(inkr) = C_n I_n(nkr) + D_n K_n(nkr),$$

where I_n is the modified Bessel function of the first kind of order n and K_n is the modified Bessel function of the second kind of order n . Of the two, I_n is finite at $r = 0$ and increases with radius, and K_n is infinite at $r = 0$ and decreases with radius. For solutions nearer the axis than the conductors, the potential thus has the form

$$V = C_n I_n(nkr) e^{in(\theta-kz)},$$

and outside the conductors, the potential has the form

$$V = D_n K_n(nkr) e^{in(\theta-kz)}.$$

At small values of it's argument, retaining only the leading terms in it's series expansion, $I_n(u) = 2^{-n} u^n/n!$. In order for the expansion of V and it's derivatives to resemble, in the limit as $k \rightarrow 0$, the expansion of a non-spiral magnet, it is necessary to include in the form for V the term $(nkr_0)^{-n}$, where r_0 is a reference radius. The interior potential is then

$$V = \sum C_n I_n(nkr)/(nkr_0)^n e^{in(\theta-kz)},$$

where the sum is from $n = 1$ to ∞ .

The asymptotic form for this as $k \rightarrow 0$ is $V = \sum C_n/(2^n n!) (r/r_0)^n e^{in\theta}$. The field is given by $\mathbf{B} = -\nabla V$; the asymptotic V gives $B_r = -\sum n C_n/(2^n n! r_0) (r/r_0)^{n-1} e^{in\theta}$ and $B_\theta = -\sum i C_n/(2^n n! r_0) (r/r_0)^{n-1} e^{in\theta} = i B_r$. Then $B_\theta + i B_r = -\sum i C_n/(2^{n-1} (n-1)!) (r/r_0)^{n-1} e^{in\theta}$, where the sum is from $n=1$ to ∞ . The usual harmonic coefficients for an untwisted magnet can be given in Cartesian coordinates as $B_y + i B_x = \sum (B_n + i A_n) e^{in\theta} (r/r_0)^n$, where the sum is from $n = 0$ to ∞ . In polar coordinates, this becomes $B_\theta + i B_r = \sum (B_n + i A_n) e^{in(\theta+1)} (r/r_0)^n$. Comparing this with the preceding, it is apparent that they are the same if $B_{n-1} + i A_{n-1} = -i C_n/(2^{n-1} (n-1)!) r_0$.

Using the complete interior potential, the field is

$$\begin{aligned} B_r &= -\sum n k C_n I_n'(nkr)/(nkr_0)^n e^{in(\theta-kz)}, \\ B_\theta &= -i \sum n C_n I_n(nkr)/(r(nkr_0)^n) e^{in(\theta-kz)}, \text{ and} \\ B_z &= i k \sum n C_n I_n(nkr)/(nkr_0)^n e^{in(\theta-kz)}, \end{aligned}$$

where the prime denotes the derivative of I_n wrt it's argument. The exterior field is

$$\begin{aligned} B_r &= -\sum n k D_n K_n'(nkr) e^{in(\theta-kz)}, \\ B_\theta &= -i \sum n D_n K_n(nkr)/r e^{in(\theta-kz)}, \text{ and} \\ B_z &= i k \sum n D_n K_n(nkr) e^{in(\theta-kz)}. \end{aligned}$$

The relation between C_n and D_n is found by assuming a current element at radius R given by \mathbf{J}

$R\Delta\theta\Delta R$, where $\mathbf{J}(R,\theta,z) = J_\theta \theta_1 + J_z \mathbf{z}_1$ is the current density in A/m^2 (θ_1 is a unit vector in the θ direction, etc.). At $r = R$, B_r is continuous, so equating the inner and outer B_r 's gives $C_n I_n'(nkR)/(nkr_0)^n = D_n K_n'(nkR)$. Then the exterior field becomes

$$\begin{aligned} B_r &= -\sum n k C_n I_n'(nkR)/[K_n'(nkR)(nkr_0)^n] K_n'(nkr) e^{in(\theta-kz)}, \\ B_\theta &= -i \sum n C_n I_n'(nkR)/[K_n'(nkR)(nkr_0)^n] K_n(nkr)/r e^{in(\theta-kz)}, \text{ and} \\ B_z &= i k \sum n C_n I_n'(nkR)/[K_n'(nkR)(nkr_0)^n] K_n(nkr) e^{in(\theta-kz)}. \end{aligned}$$

The constants C_n can be determined for the current element by considering a closed path in the $z = \text{constant}$ plane enclosing the element at radius R . Applying Ampere's circuital law gives $(B_{\theta,\text{out}} - B_{\theta,\text{in}}) \big|_{r=R} = \mu_0 J_z \Delta R$, or

$$J_z = -i/(\mu_0 k R^2 \Delta R) \sum C_n e^{in(\theta-kz)}/[(nkr_0)^n K_n'(nkR)],$$

where use has been made of the Wronskian $I_n(u)K_n'(u) - K_n(u)I_n'(u) = -1/u$. Similarly, considering a closed path at constant radius enclosing the current element, $(B_{z,\text{in}} - B_{z,\text{out}}) \big|_{r=R} = \mu_0 J_\theta \Delta R$, so

$$J_\theta = -i/(\mu_0 R \Delta R) \sum C_n e^{in(\theta-kz)}/[(nkr_0)^n K_n'(nkR)].$$

It may be noted that $J_\theta/J_z = kR = 2\pi R/L$, which says that \mathbf{J} is in the direction of the helix; the helix angle is $\tan^{-1}(2\pi R/L)$.

The C_n may be obtained in terms of either current density component; if the equation for J_z is multiplied through by $e^{-im\theta}$ and integrated wrt θ over 2π , $\int e^{i(n-m)\theta} d\theta$ is zero if $n \neq m$, and 2π if $n=m$. This gives

$$C_n = i(\mu_0/2\pi)kR^2(nkr_0)^n K_n'(nkR)\Delta R \int J_z e^{-in(\theta-kz)} d\theta.$$

If the winding is thick, this may be integrated wrt R over the radial extent of the winding.

Caspi's treatment [2] allows for a sum over submultiples of the pitch length, i.e., $k_m = 2\pi m/L$, $m = 1$ to ∞ ; this complete solution permits one to give a correct description of the field in the ends of a helical magnet. For the present purpose, it is sufficient to integrate the C_n expression above wrt z throughout the end, giving a form of "unit-meters"; from this the effective length of a harmonic is obtained by dividing by the corresponding C_n for the helical section of the magnet.

References

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