Basics of Polarized Beam Acceleration

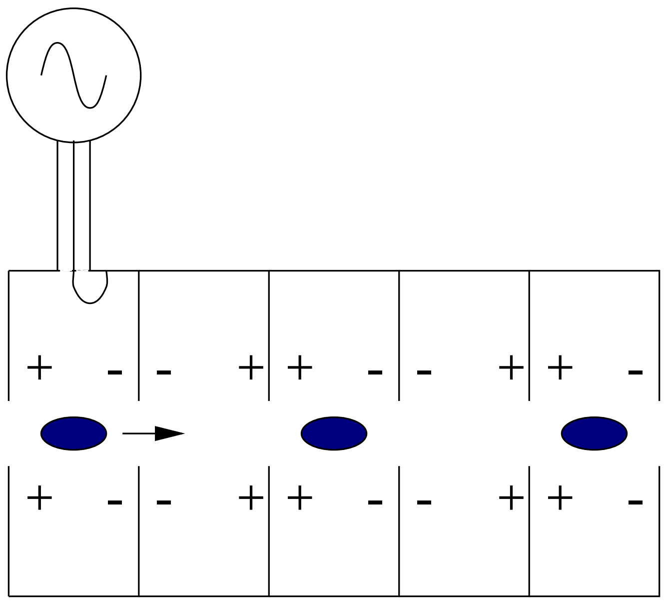
Protons:

- Transverse beam dynamics.
- Simple model of the proton.
    - Spin dynamics.
    - Depolarizing resonances.
    - Siberian snakes.
- The real machines: RHIC and injectors.

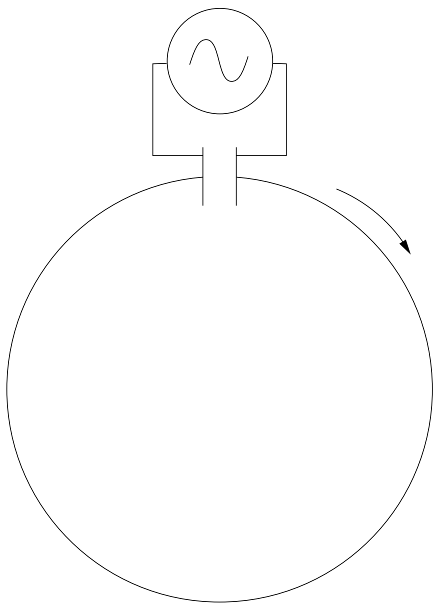
Electrons/Positrons:

- Longitudinal beam dynamics.
    - Synchrotron oscillations and tune.
    - Electrons: Synchrotron radiation
- Radiative polarization.
- Quantum fluctuations ⇒ Spin Diffusion
- Polarization in some real $e^{\pm}$ machines.
- Measurements with polarized $e^{\pm}$ beams.

Acceleration with RF cavities



Linac: $\vec{F} = q\vec{E}(t)$.

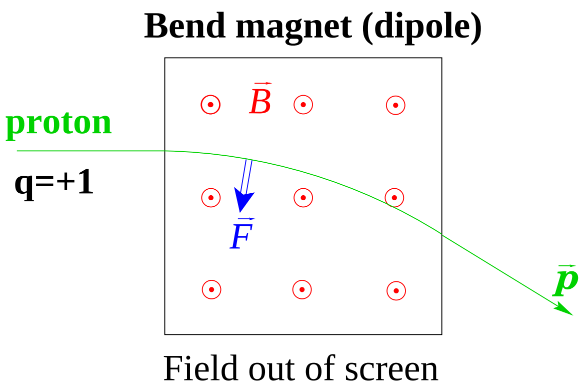


Ring with rf cavity

- Must maintain synchronism of bunch with rf phase.
- Particles oscillate in energy about the stable synchronous phase.

Particle Trajectories in Magnetic Fields

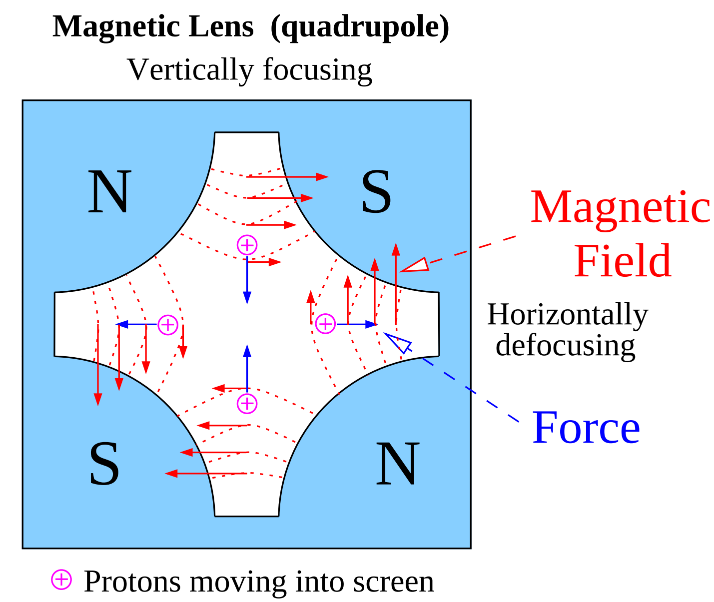
Dipole magnets bend the beam around the ring.



Charged particles are deflected by magnetic fields. Lorentz Force:

$$\vec{F} = \frac{q}{\gamma m}\vec{p} \times \vec{B}$$

Quadrupole magnets focus the beam for stability.



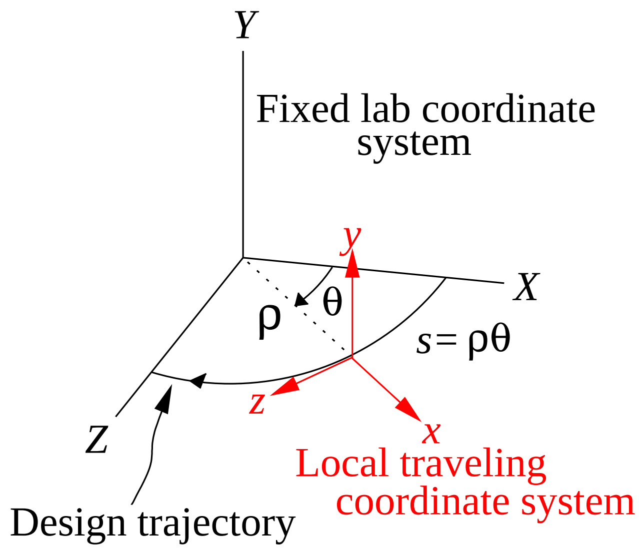
BROOKHAVEN NATIONAL LABORATORY

Hamiltonian without Spin

$$H(X, P_X, Y, P_Y, Z, P_Z; t) = \sqrt{(\vec{P} - q\vec{A})^2 + m^2c^4} + q\phi$$

After a bunch of canonical transformations and $\phi = 0$, $\vec{A} = (0, 0, A_s)$:

$$\mathcal{H}(x, x', y, y', z, \delta p/p_0; s) \simeq -\frac{q}{p_0}A_s - \left(1 + \frac{x}{\rho}\right)\left(1 + \frac{\delta p}{p_0} - \frac{1}{2}(x'^2 + y'^2) + \cdots\right)$$



$$\rho = \frac{p}{qB_\perp}$$

$$x' = \frac{dx}{ds}$$

$$y' = \frac{dy}{ds}$$

Paraxial approx.: $|x'|, |y'| \ll 1$

Hill's Equations

$$x'' + k_x(s)x = \frac{\delta}{\rho(s)},$$
$$y'' + k_y(s)y = 0,$$
$$\text{with} \quad \delta = \frac{\delta p}{p_0}.$$

For quadrupoles:

$$k_x = \frac{q}{p}\frac{\partial B_y}{\partial x}$$
$$k_y = -\frac{q}{p}\frac{\partial B_y}{\partial x}$$

Harmonic oscillator with periodic spring constant.

Periodic conditions: $k_j(s+L) = k_j(s), \qquad \rho(s+L) = \rho(s)$

where L is length of periodic cell.

- Horizontal motion has inhomogeneous dispersion term.
  - Ignore it for now.

Solutions to Hill's Equation

Use Floquet's (Block's) Theorem $\Rightarrow$
Quasi-periodic solutions of form:

$$x(s) = \sqrt{\mathcal{W}\beta(s)}\,\cos(\psi(s)), \quad \text{with}$$
$$\psi'(s) = \frac{1}{\beta(s)}.$$

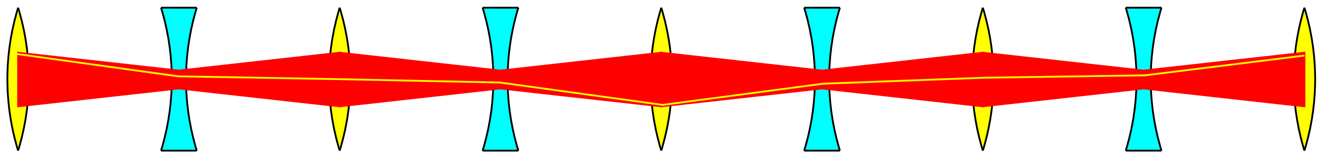
Periodicity of β-function: $\beta(s+L) = \beta(s)$.

Note: In general $\psi(s+L) \neq \psi(s) + n2\pi$. Resonances are bad!

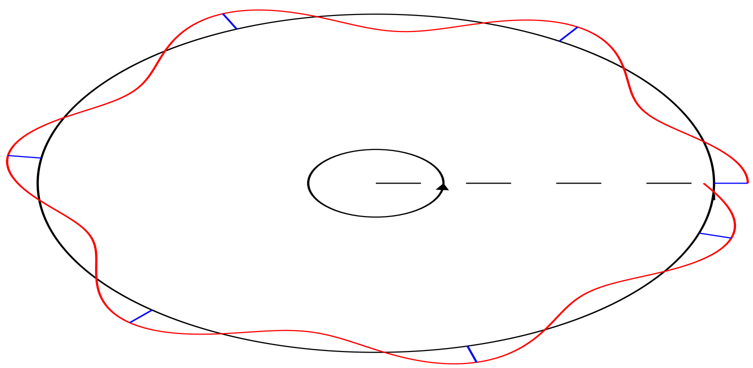
$$x'(s) = -\sqrt{\frac{\mathcal{W}}{\beta}}\left(\alpha\cos\psi + \sin\psi\right),$$

with $\alpha = -\frac{1}{2}\beta'$.

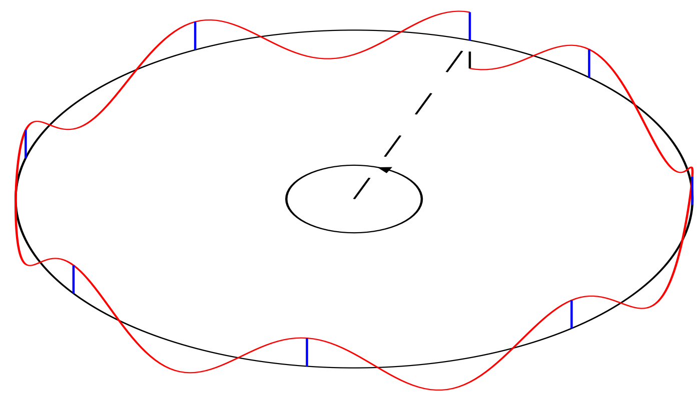
Transport and Betatron Oscillations



Alternate focusing and defocusing lenses for stability.



Horizontal Betatron Oscillation
with tune: $Q_h = 6.3$,
i.e., 6.3 oscillations per turn.



Vertical Betatron Oscillation
with tune: $Q_v = 7.5$,
i.e., 7.5 oscillations per turn.

Courant–Snyder Invariant

For a particular trajectory with initial conditions:

- Solve for $\sin\psi$ and $\cos\psi$ from equations for x and x'.
- Use $\sin^2\psi + \cos^2\psi = 1$ to get an invariant:

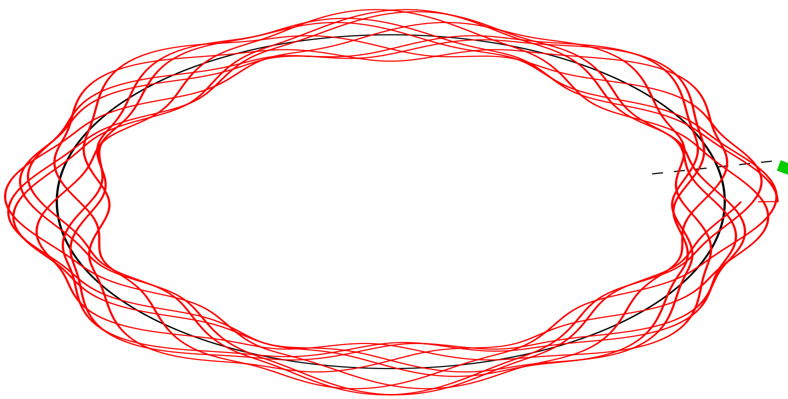
$$\mathcal{W} = \frac{1}{\beta}\left[y^2 + (\alpha y + \beta y')^2\right] \tag{1}$$

- Functions of s: $y(s)$, $y'(s)$, $\beta(s)$, $\alpha(s)$. (β and α are periodic.)
- Eq. (1) is the equation for an ellipse.
  - Area of ellipse $= \pi\mathcal{W}$.
- Beam envelope: $\pm\sqrt{\beta(s)\epsilon}$
  - $\pi\epsilon$ is the rms emittance

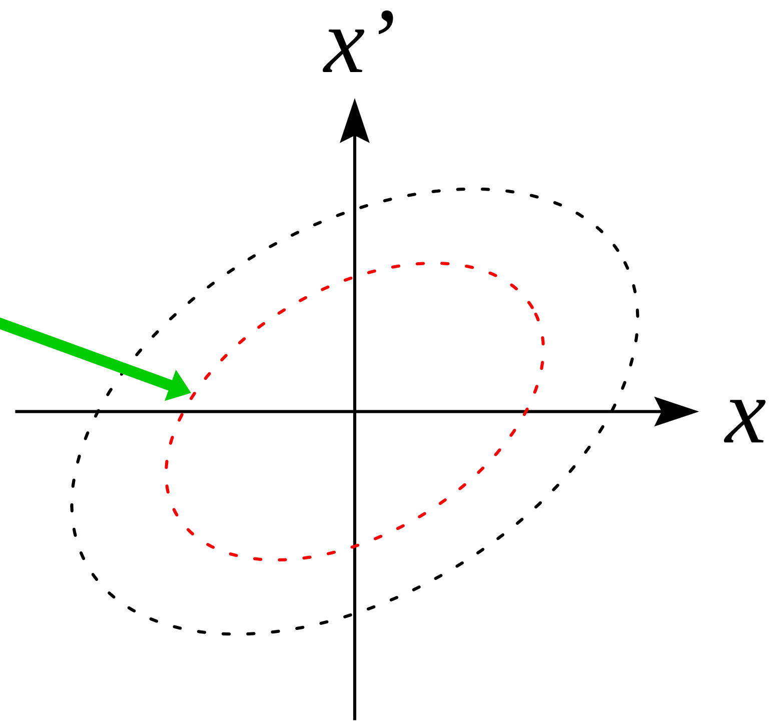
Liouville's Theorem

- Most beams have a low enough density, so that we ignore hard collisions between particles.
  - Thus we can use a 6d phase space rather than a 6N-d phase space.
- In the phase space of coordinates and their corresponding canonical momenta, the phase flow of the particle trajectories evolves so that the volumes of differential volume elements are preserved.
  - In other words, the Jacobian determinant is 1.
- Emittance is the area of the projection of the beam's phase-space volume onto a particular (x_i, P_i) plane.

2d Phase Space Plots

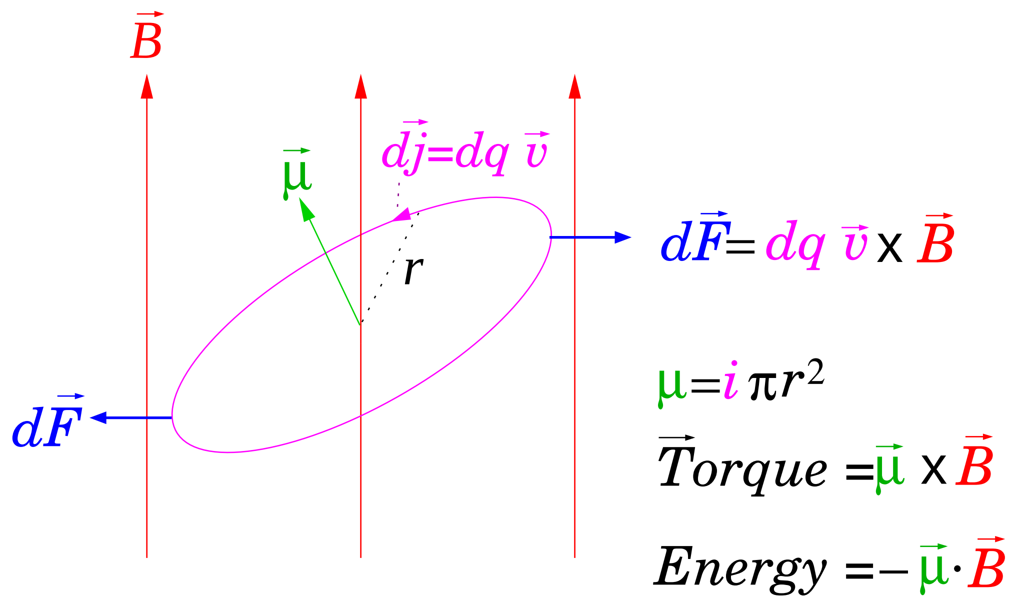


Horizontal Betatron Oscillation
with tune: $Q_x = 3.28$,
tracked through 10 turns
with 8 periodic cells.



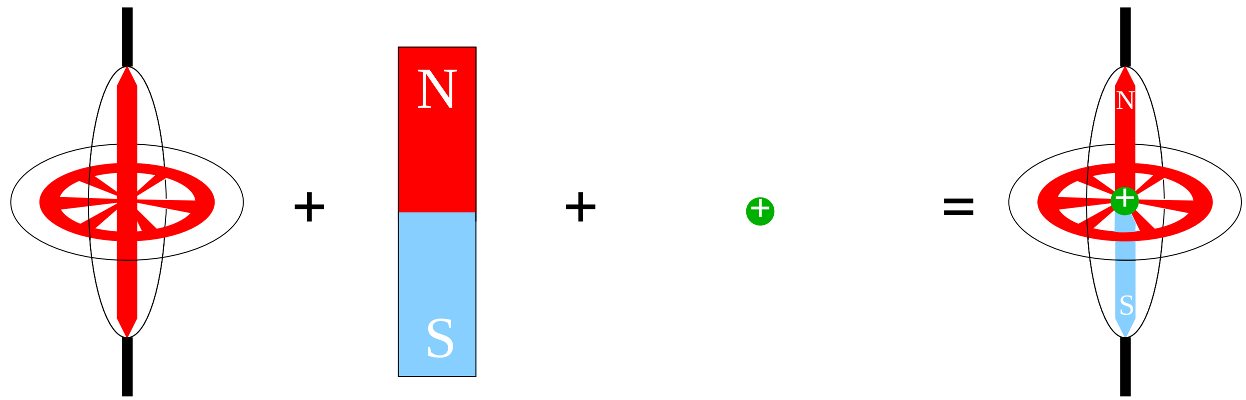
Poincaré plot of proton on successive turns for one location in the ring.

Torque on Classical Magnetic Moment



- Semiclassical model:
  - The spin $\vec{S}$ has a constant magnitude in the rest frame.
  - The magnetic moment $\vec{\mu} \propto \vec{S}$.
    - $\vec{\mu}$ has a constant magnitude in the rest frame.
      (Sort of like a loop of infinite inductance.)

Simple Model of Proton



Gyroscope + Bar magnet + Charge = "proton"

Spin	Magnetic Dipole Moment

Polarization: Average spin of the ensemble of protons.

$$\vec{P} = \frac{1}{N}\sum_{j=1}^{N}\frac{\vec{S}}{|S|}$$

Relativistic Angular Momentum

Energy-momentum tensor (à la Weinberg)

$$T^{\alpha\beta}(x) = T^{\beta\alpha}(x) = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n}\delta^3(x - x_n(t))$$

For isolated system

$$\frac{\partial}{\partial x^\alpha}T^{\alpha\beta} = 0.$$

Define 4d analogue of $\vec{r}\times\vec{p}$:

$$M^{\alpha\beta\gamma} = x^\alpha T^{\beta\gamma} - x^\beta T^{\alpha\gamma}$$

$$J^{\alpha\beta} = \int M^{0\alpha\beta} d^3x = \int x^\alpha T^{\beta 0} - x^\beta T^{\alpha 0} d^3x$$

Spin (intrinsic angular momentum):

$$S_\alpha = \tfrac{1}{2c}\epsilon_{\alpha\beta\gamma\delta}J^{\beta\gamma}u^\delta, \qquad \text{proper velocity: } u^\delta = \tfrac{dx^\delta}{d\tau}.$$

For a particle at rest with CM at rest at the origin:

$$J^{\diamond\mu\nu} : \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & S_z^\diamond & -S_y^\diamond \\ 0 & -S_z^\diamond & 0 & S_x^\diamond \\ 0 & S_y^\diamond & -S_x^\diamond & 0 \end{pmatrix}, \qquad (\vec{J}^\diamond = \vec{S}^\diamond)$$

Boost along z:

$$J^{\mu\nu} : \begin{pmatrix} 0 & \gamma\beta S_y^\diamond & -\gamma\beta S_x^\diamond & 0 \\ -\gamma\beta S_y^\diamond & 0 & S_z^\diamond & -\gamma S_y^\diamond \\ \gamma\beta S_x^\diamond & -S_z^\diamond & 0 & \gamma S_x^\diamond \\ 0 & \gamma S_y^\diamond & -\gamma S_x^\diamond & 0 \end{pmatrix}, \quad \Rightarrow \quad \vec{J} = \begin{pmatrix} \gamma S_x^\diamond \\ \gamma S_y^\diamond \\ S_z^\diamond \end{pmatrix}$$

$$S^\mu : \begin{pmatrix} \gamma\beta S_z^\diamond \\ S_x^\diamond \\ S_y^\diamond \\ \gamma S_z^\diamond \end{pmatrix}, \quad \Rightarrow \quad \vec{S} = \begin{pmatrix} S_x^\diamond \\ S_y^\diamond \\ \gamma S_z^\diamond \end{pmatrix}, \quad S^0 = \vec{\beta}\cdot\vec{S}$$

$$\vec{J} - \vec{S} = \begin{pmatrix} (\gamma-1)S_x^\diamond \\ (\gamma-1)S_y^\diamond \\ (1-\gamma)S_z^\diamond \end{pmatrix}$$

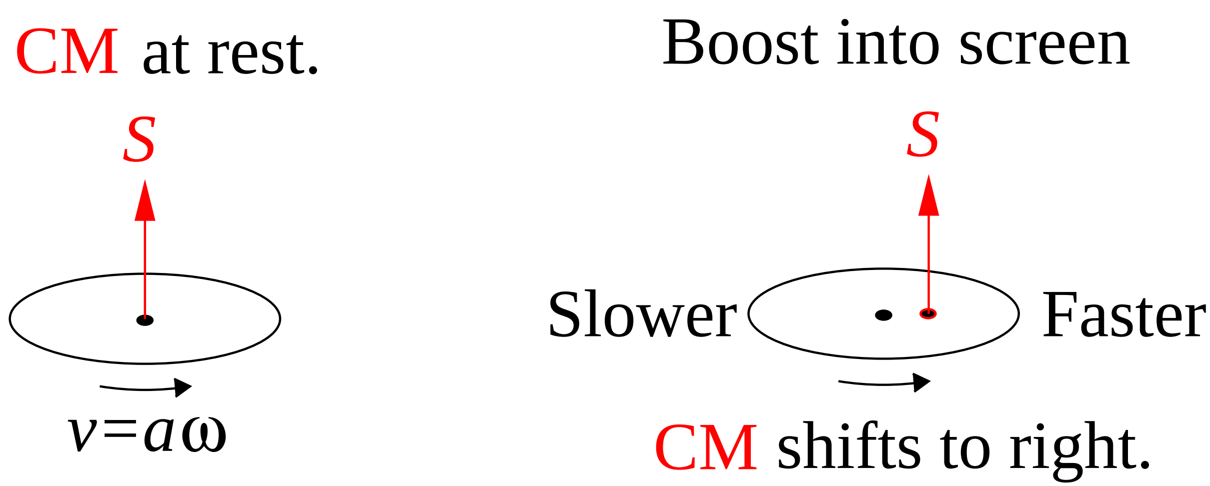
Center-of-Mass shift

$$\vec{r}_{\mathrm{CM}} \times \vec{p}_{\mathrm{CM}} = (\vec{J} - \vec{S})_\perp$$

$$\gamma\beta mc(-x_{\mathrm{CM}}\,\hat{y} + y_{\mathrm{CM}}\,\hat{x}) = (\gamma - 1)\vec{S}^{\diamond}_{\perp}$$

$$\gamma\beta mc(\vec{x}_{\mathrm{CM}} + \vec{y}_{\mathrm{CM}}) = (\gamma - 1)\,\hat{z} \times \vec{S}^{\diamond}_{\perp}$$

$$\vec{r}_{\perp\,\mathrm{CM}} = \frac{\gamma}{\gamma + 1}\,\frac{\vec{\beta} \times \vec{S}}{mc}$$

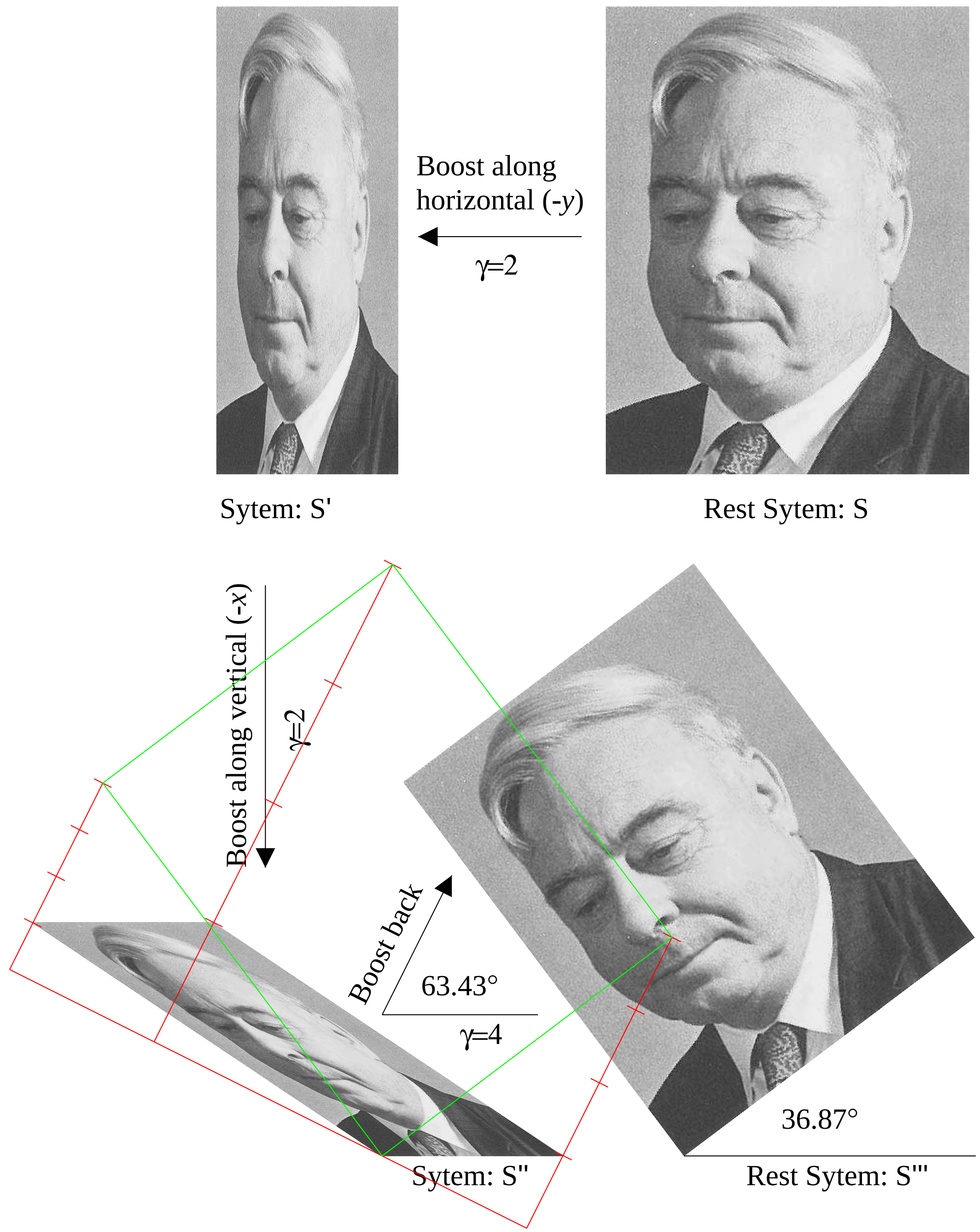


Center of charge wobbles: classical "Zitterbewegung"

Thomas Precession

1. Boost observer to left.
2. Boost observer downward.
3. Boost back to rest.

- Net rotation of rest frame.



Thomas—Frenkel (BMT) Equation

In the local rest frame of the proton, the spin precession of the proton obeys the Thomas-Frenkel equation:

$$\frac{d\vec{S}^\diamond}{dt} = \frac{q}{\gamma m}\vec{S}^\diamond \times \left[(1+G\gamma)\vec{B}_\perp + (1+G)\vec{B}_\parallel + \left(G\gamma + \frac{\gamma}{\gamma+1}\right)\frac{\vec{E}\times\vec{v}}{c^2}\right].$$

This is a mixed description: t, $\vec{B}$, and $\vec{E}$ in the lab frame, but spin $\vec{S}^\diamond$ in local rest frame of the particle:

Proton:	$G = \dfrac{g-2}{2} = 1.792847,$	523.34 MeV/unit $G\gamma$
Electron:	$a = G = \dfrac{g-2}{2} = 0.001159652,$	440.65 MeV/unit $a\gamma$

$$\gamma = \frac{\text{Energy}}{mc^2}.$$

Thomas—Frenkel (BMT) Equation

In the local rest frame of the proton, the spin precession of the proton obeys the Thomas-Frenkel equation:

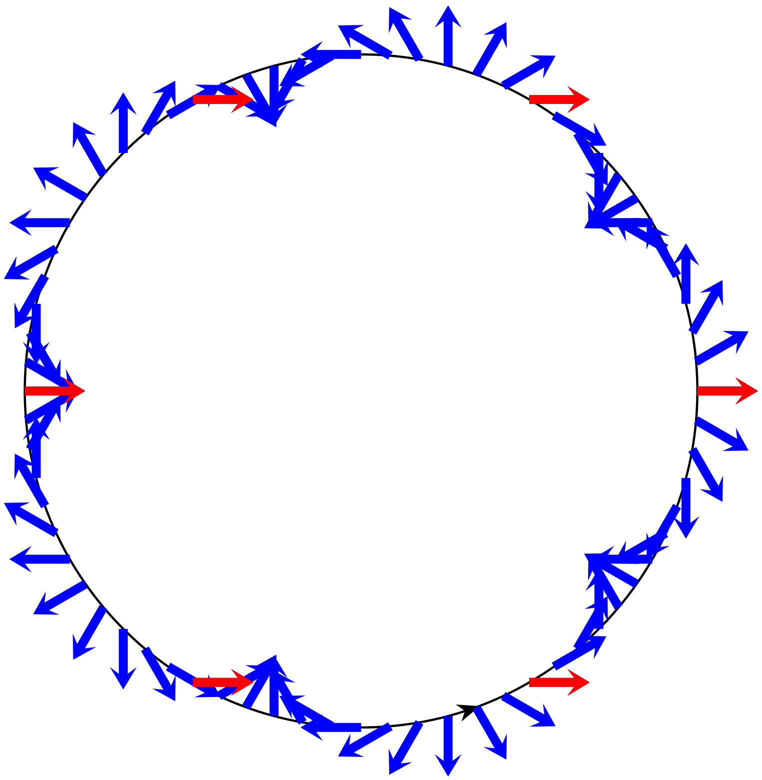
$$\text{Torque}: \quad \frac{d\vec{S}^\diamond}{dt} = \frac{q}{\gamma m}\vec{S}^\diamond \times \left[(1+G\gamma)\vec{B}_\perp + (1+G)\vec{B}_\parallel\right] \qquad \text{TF}$$

$$\text{Force}: \quad \frac{d\vec{p}}{dt} = \frac{q}{\gamma m}\vec{p} \times \vec{B}_\perp \qquad \text{Lorentz}$$

(This is a mixed description: t, and $\vec{B}$ in the lab frame, but spin $\vec{S}^\diamond$ in local rest frame of the proton.)

$$G = \frac{g-2}{2} = 1.7928, \qquad \gamma = \frac{\text{Energy}}{mc^2}.$$

Spin Precession in a Ring



Example with 6 precessions of spin in one turn:

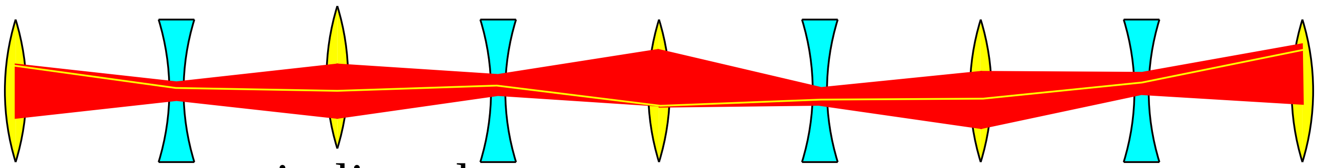
$$G\gamma + 1 = 6.$$

Spin tune: number of precessions per turn relative to beam's direction.
So we subtract one:

$$\nu_{\text{spin}} = G\gamma \quad \propto \quad \text{energy},$$

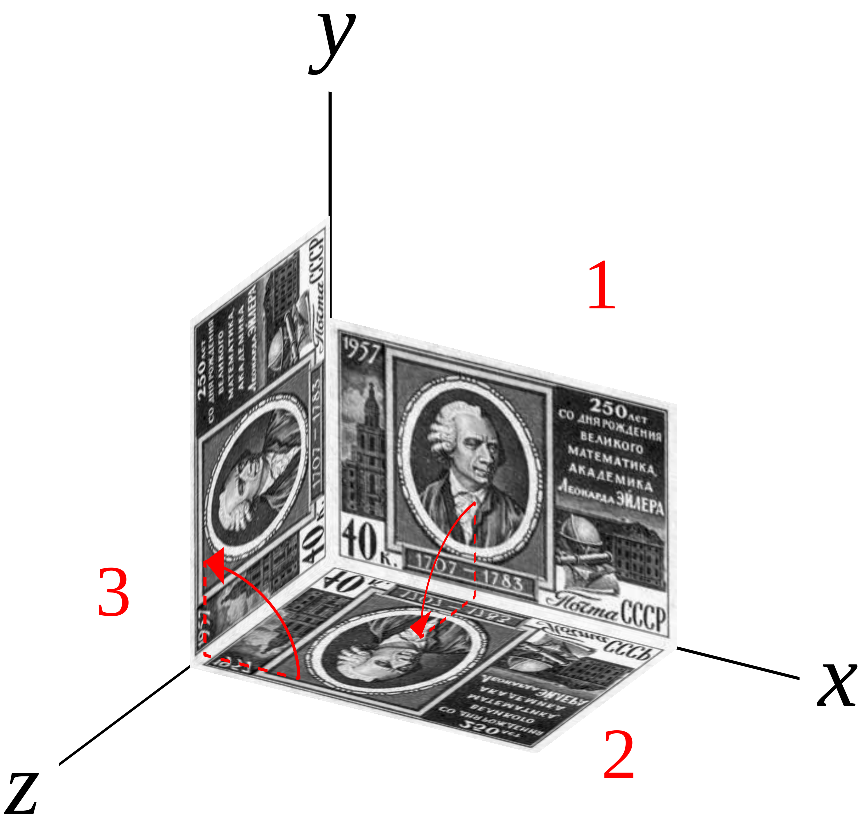
i.e., 5 in this example.

Misalignments or Imperfections

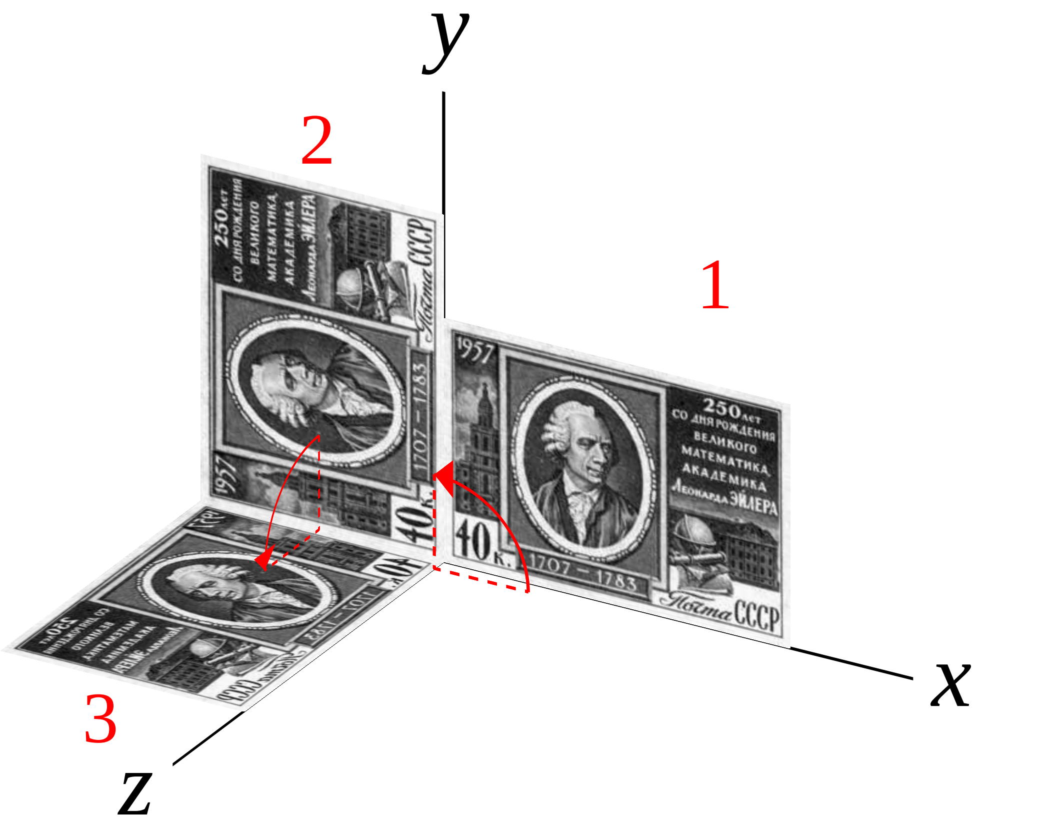


- A misaligned quadrupole creates a steering error which propagates through the lattice.
- For an accelerator ring, this shifts the closed orbit away from the design trajectory.
- If the misalignment is vertical, then the design trajectory will have a periodic set of small vertical bends interspersed with the normal horizontal bends of the bending magnets.
- This leads to an integer resonance condition for the spin tune.

In general, rotations don't commute.



$R_z(90°)\, R_x(-90°)$



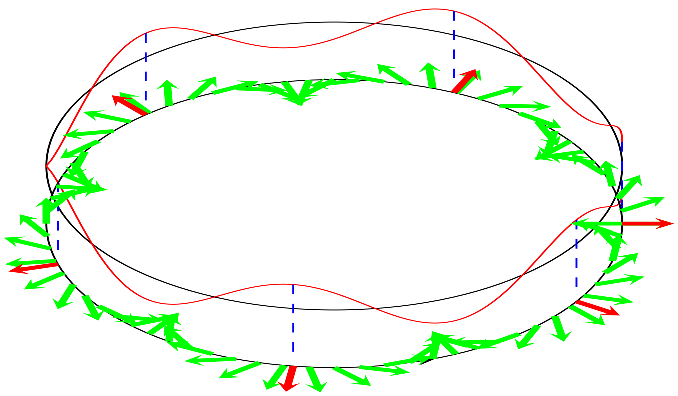
$R_x(-90°)\, R_z(90°)$

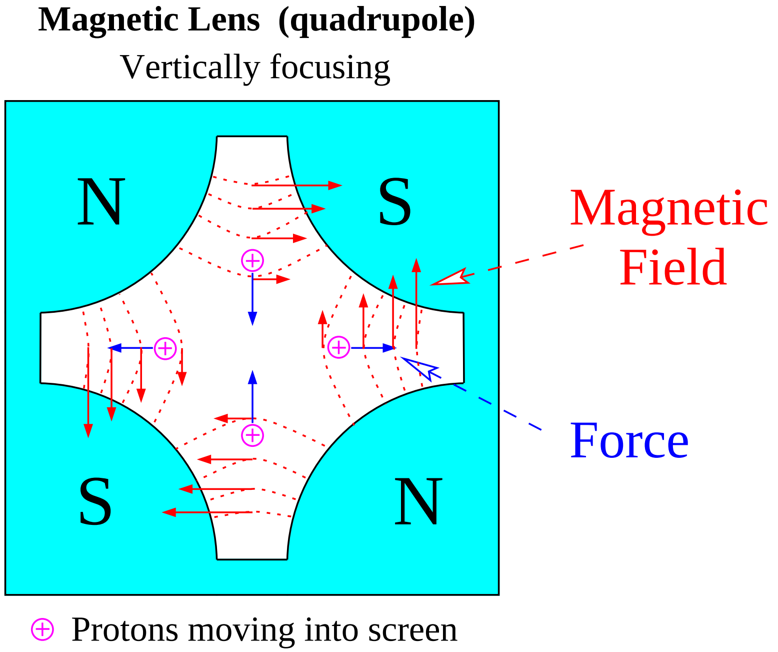
Depolarizing Resonances

Simple Resonance Condition:

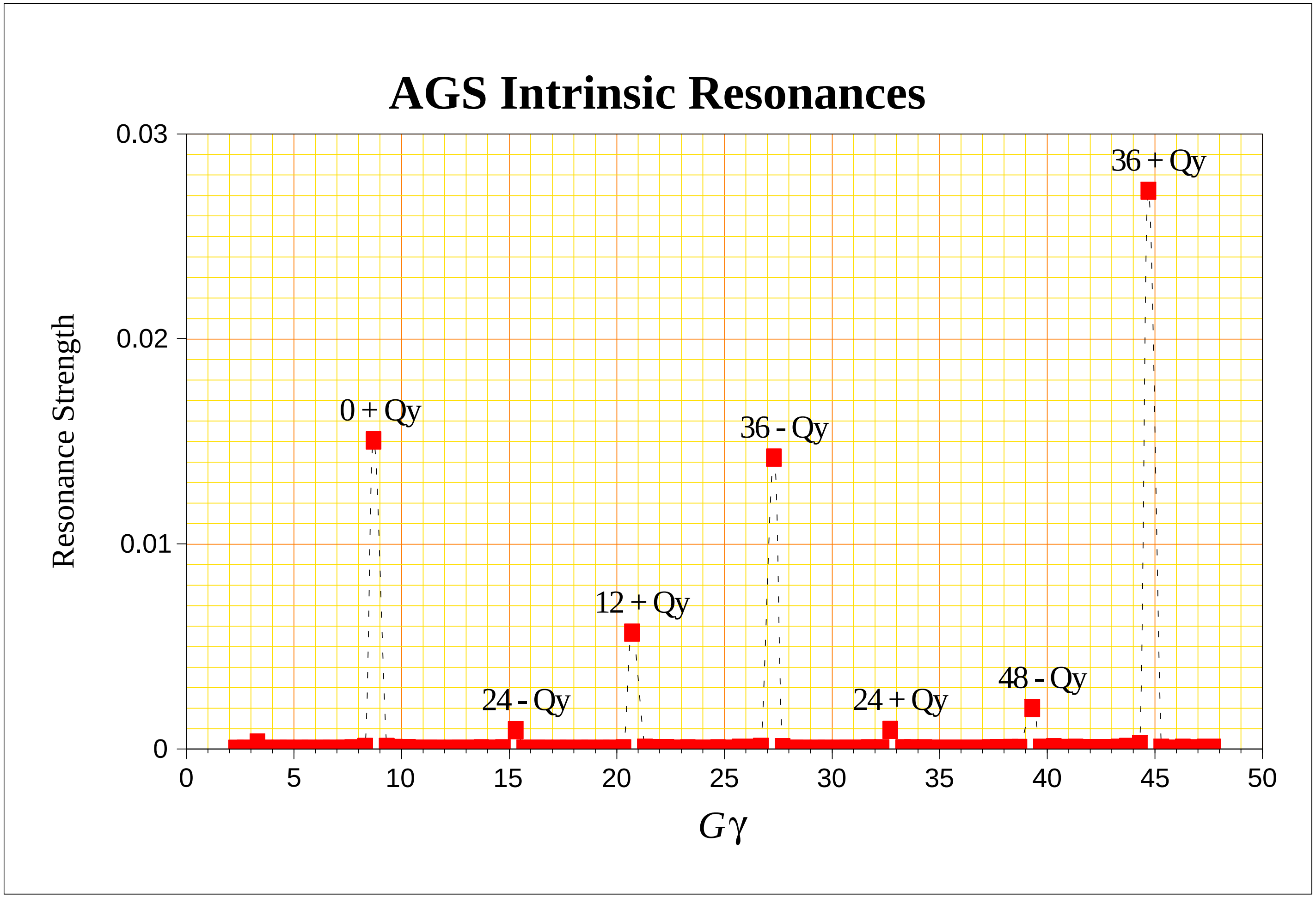
$$\nu_{\text{spin}} = \underset{\text{(imperfection)}}{N} + \underset{\text{(intrinsic)}}{N_{\text{v}} Q_{\text{v}}},$$

where N and N_{v} are integers.





AGS Intrinsic Resonances



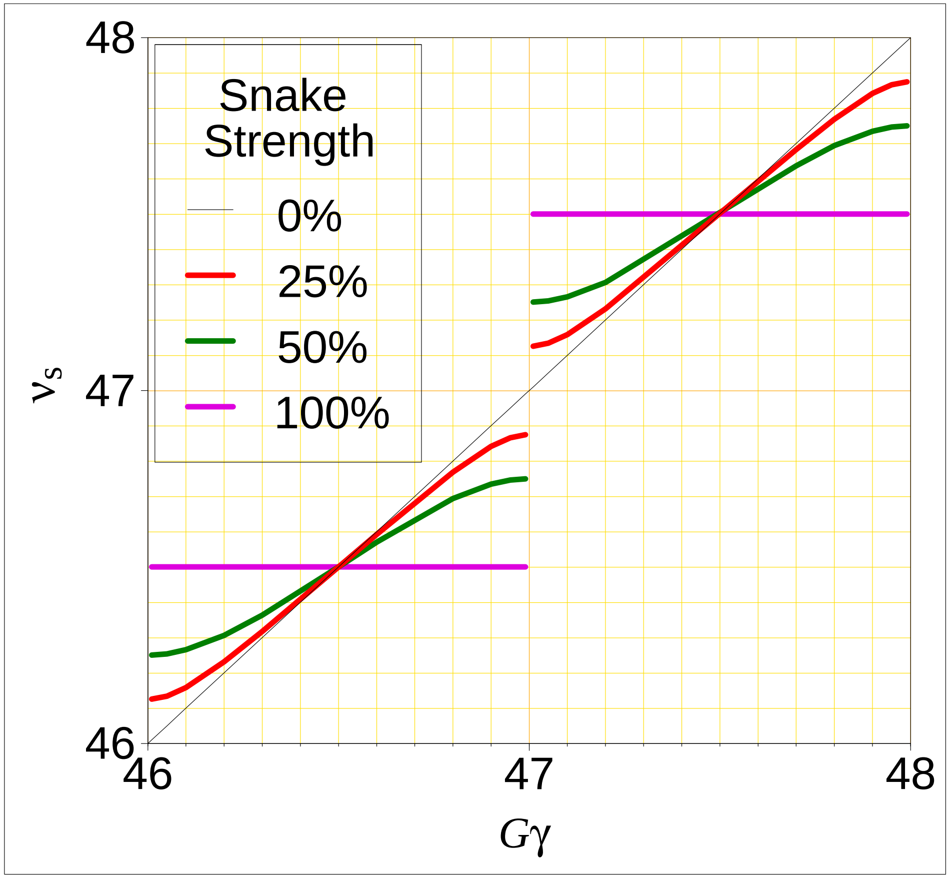
Partial Snakes

Adding a partial snake opens up stop bands around the integer imperfection resonances.

At the snake the stable spin direction points along the snake's rotation axis when $G\gamma =$ integer.

Partial snake strength: $\frac{\mu}{\pi}$

$$\cos \pi\nu_{\rm s} = \cos(G\gamma\pi)\cos\frac{\mu}{2}$$



BROOKHAVEN NATIONAL LABORATORY

Crossing an Isolated Spin Resonance

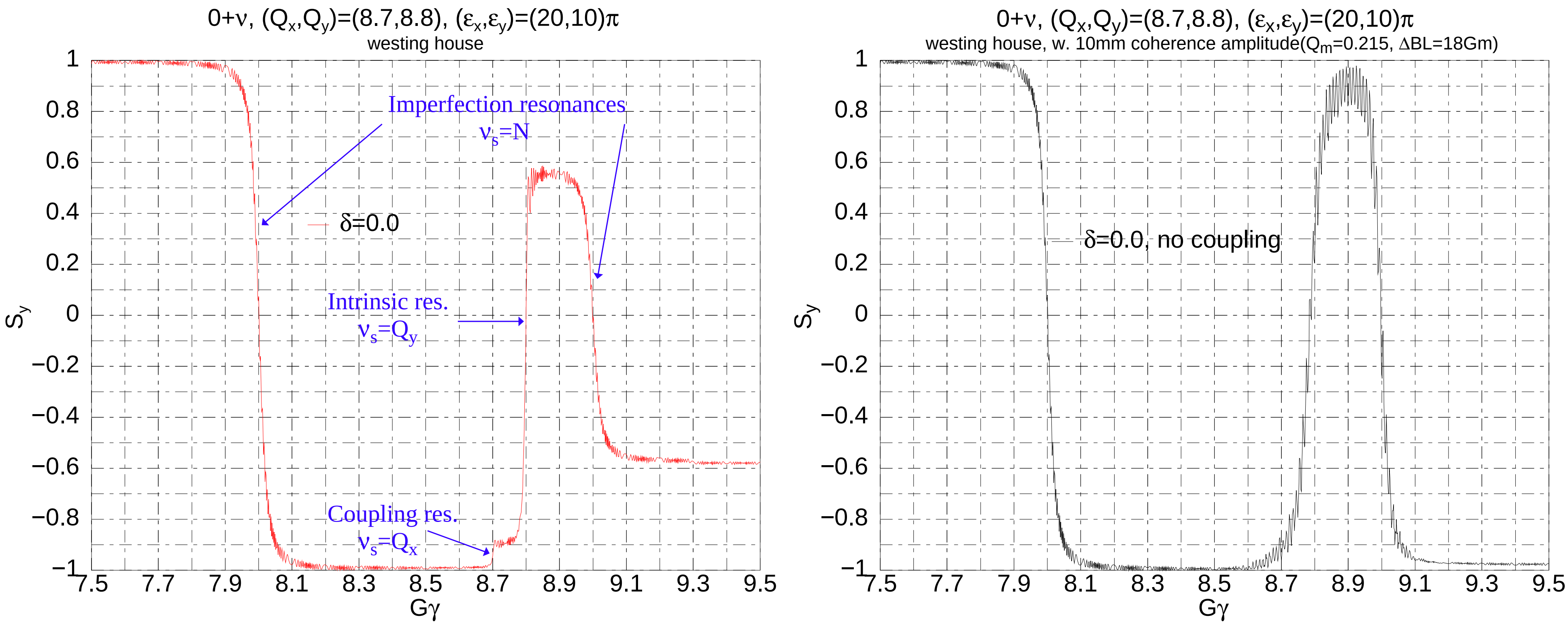
Froissart—Stora Formula:

$$\frac{P_\mathrm{f}}{P_\mathrm{i}} = 2\exp\left(-\frac{\pi|\epsilon|^2}{2\alpha}\right) - 1.$$

Ramp rate: $\alpha = \frac{dG\gamma}{d\theta}$, $(\theta:\ 2\pi/\mathrm{turn.})$

Resonance strength: $\epsilon =$Fourier amplitude.

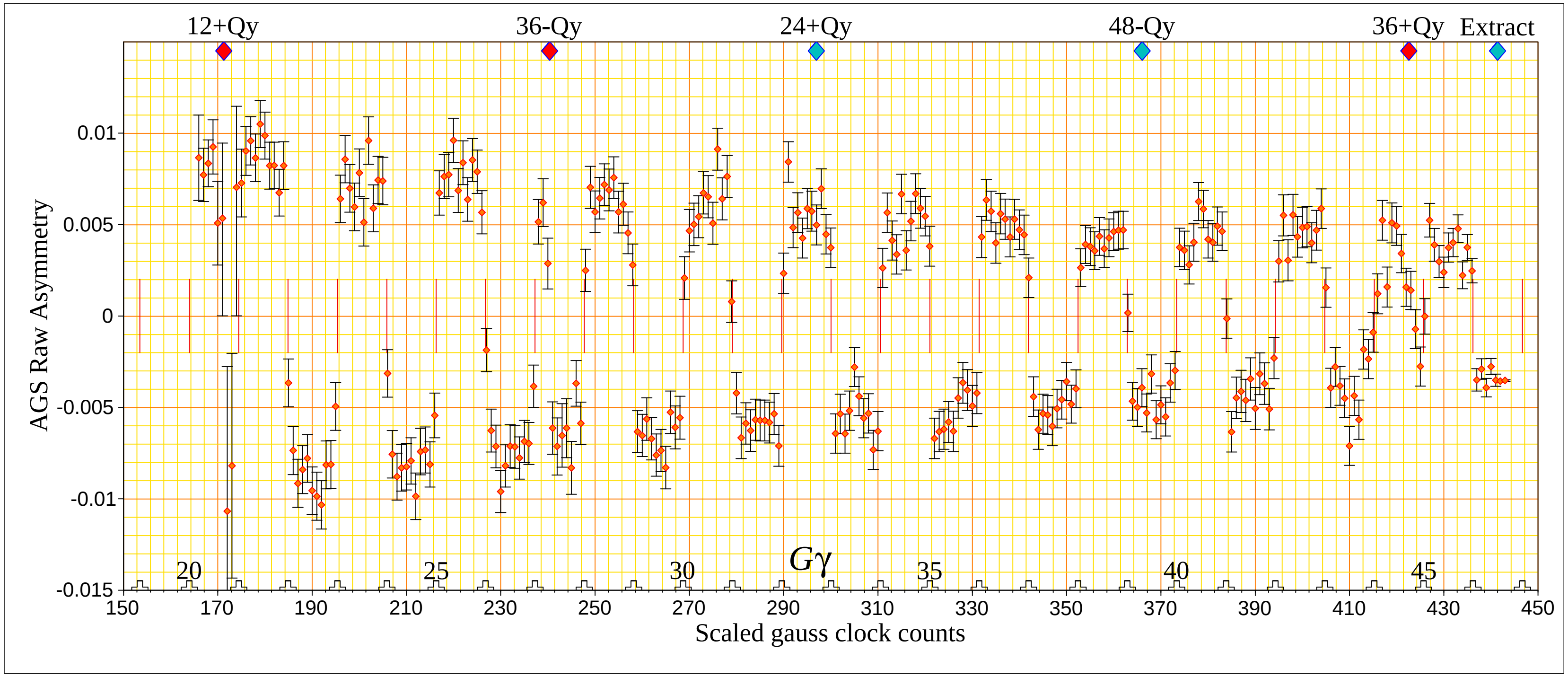
Resonance Crossing in AGS



AC dipole used to increase strength of $\nu_{\mathrm{s}} = Q_y$ resonance.

(Simulations by Mei Bai)

AGS Raw Asymmetry during Ramp



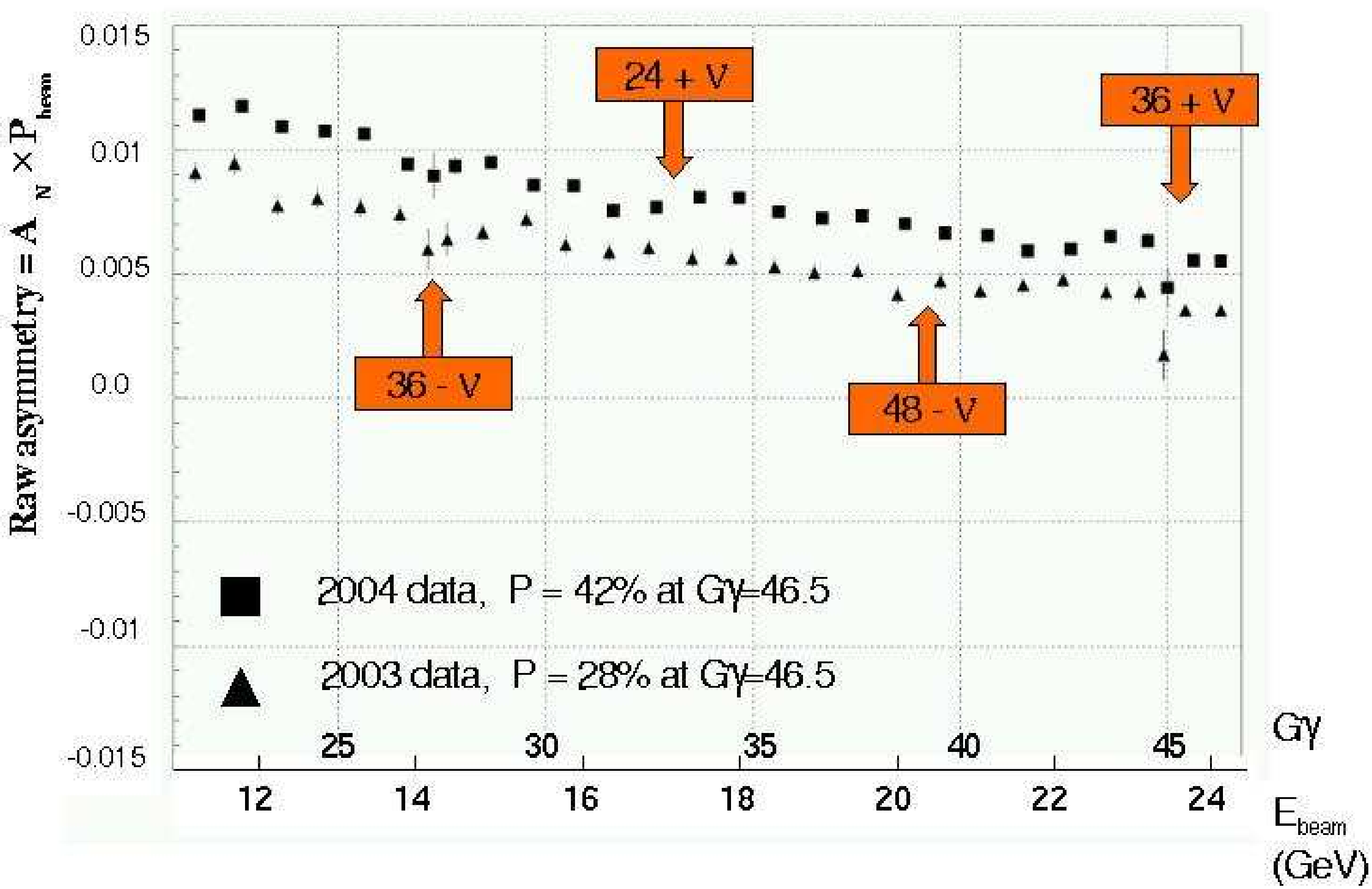
AGS has 12 superperiods.
Vertical betatron tune: 8.7
Snake strength: 5%

(From Jeff Woods)

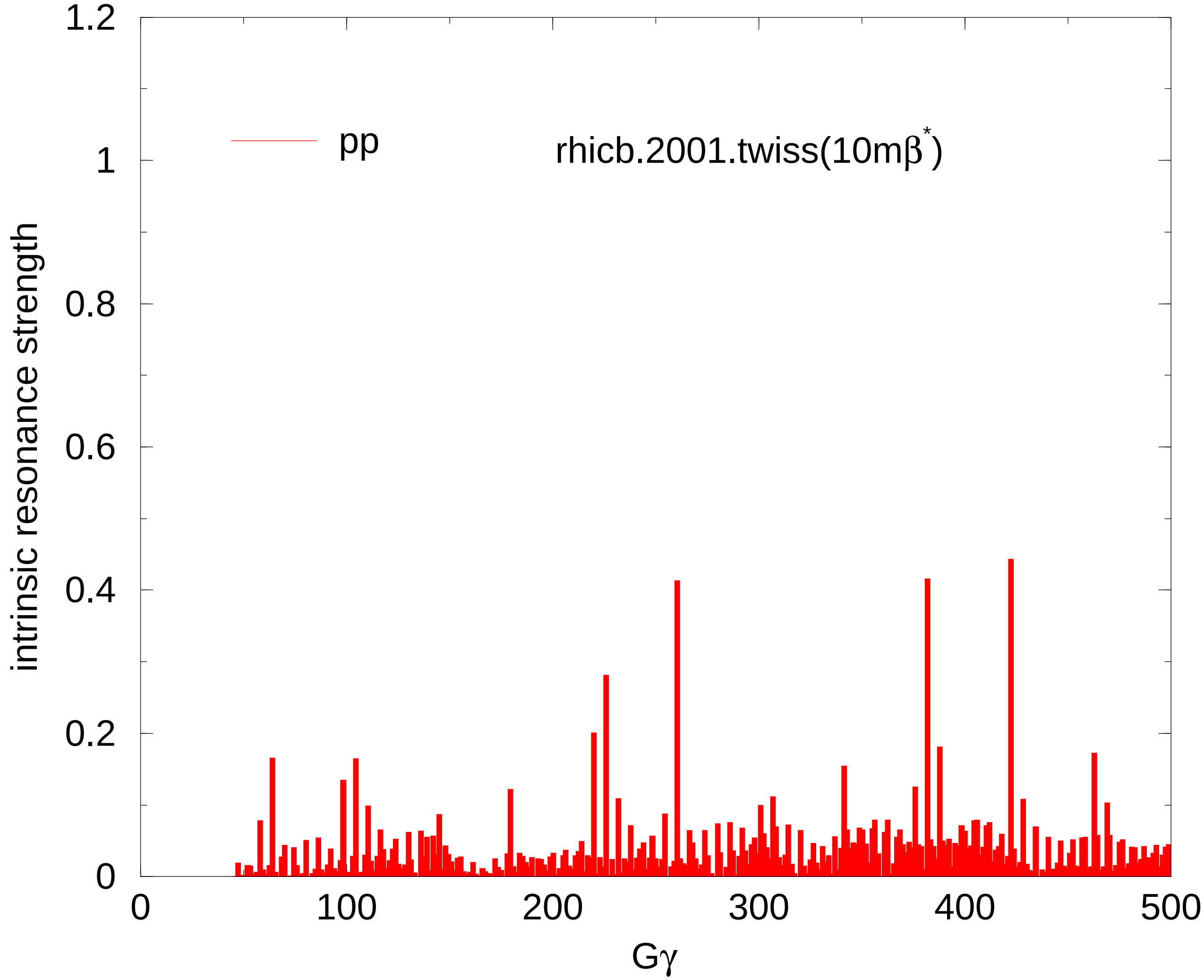
AC dipole pulses at resonances:

- $0 + Q_y$
- $12 + Q_y$
- $36 - Q_y$
- $36 + Q_y$

pC CNI Asymmetry during AGS Ramp



Depolarizing Resonances in RHIC



$$Q_x = 28.236$$

$$Q_y = 29.219$$

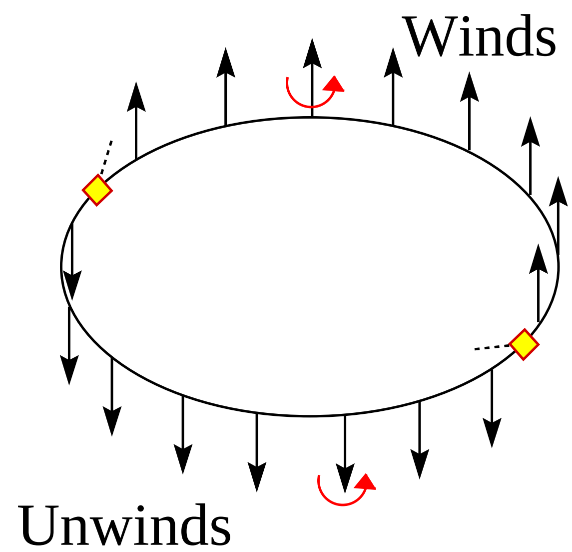
$$\pi\epsilon_y = 10\pi\ \mu\mathrm{m}$$

Will depolarize beam during acceleration.

Solution: Snakes

Snake Charming

- 2 snakes: spin is up in one half of the ring, and down in the other half.
- Spin tune: $\nu_{\text{spin}} = \frac{1}{2}$
  (It's energy independent.)
- "The unwanted precession which happens to the spin in one half of the ring is unwound in the other half."



Hamiltonian with Spin

(Here I drop the "$\diamond$" superscript on $\vec{S}$.)

$$\begin{aligned} \frac{d\vec{S}}{dt} &= \vec{W} \times \vec{S} \\ H(\vec{q}, \vec{P}, \vec{S}; s) &= \mathcal{H}_{\rm orb} + \mathcal{H}_{\rm spin} \\ &= \mathcal{H}_{\rm orb} + \vec{W} \cdot \vec{S} + O(\hbar^2) \end{aligned}$$

Group symmetries:

- Orbital motion without spin: Sp(6,r).
- Spin by itself: $\mathrm{SU}(2,\mathrm{c}) \cong \mathrm{SO}(3,\mathrm{r})$ (homomorphic).
- Full blown symmetry: $\mathrm{Sp}(6,\mathrm{r}) \oplus \mathrm{SU}(2,\mathrm{c})$.
  - Spin dependence on orbit (Thomas-Frenkel).
  - Orbit dependence on spin (Stern-Gerlach Force)—Usually ignored.

Representation of Rotations

SU(2) with usual spinor notation:

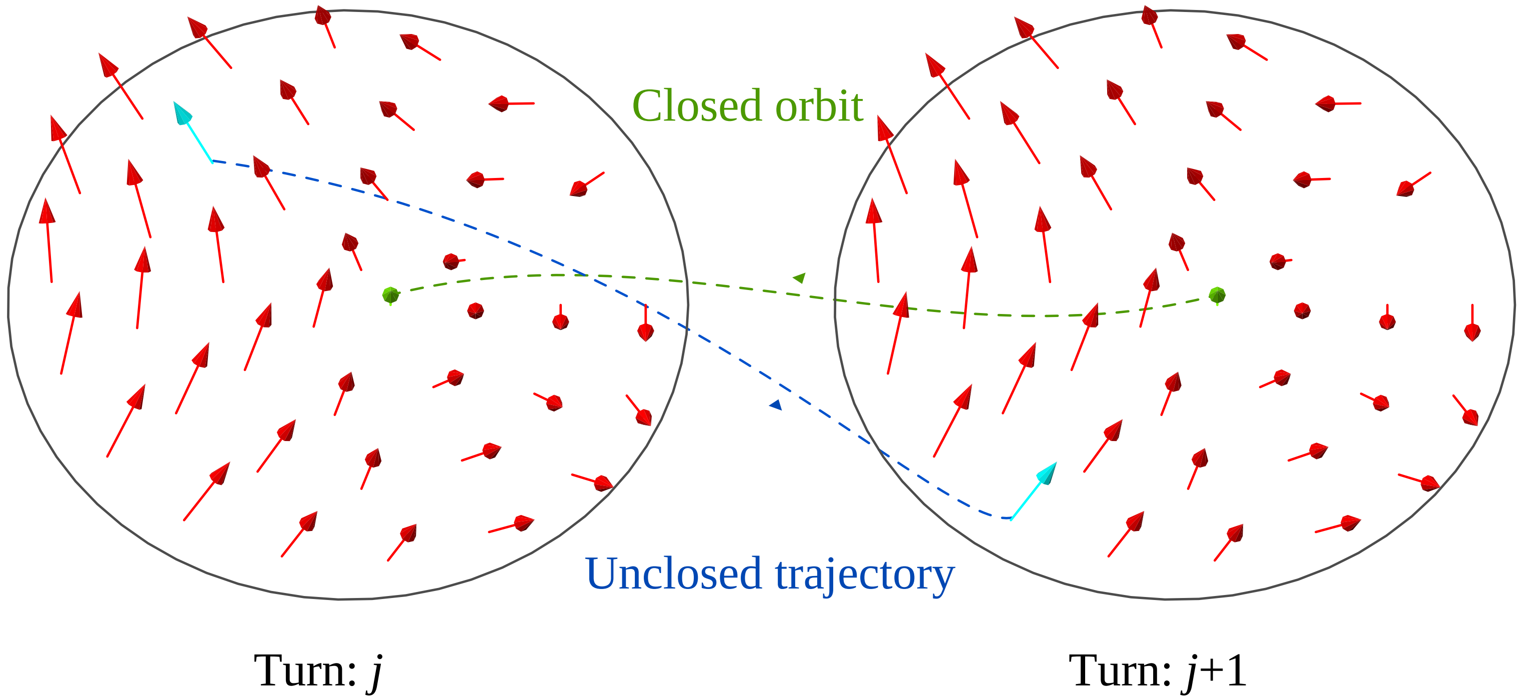
Pauli matrices: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\mathbf{R}_{\hat{n}}(\theta) = e^{i\,\hat{n}\cdot\vec{\sigma}\,\theta/2} = \begin{pmatrix} \cos\frac{\theta}{2} + i n_z \sin\frac{\theta}{2} & (n_y + i n_x)\sin\frac{\theta}{2} \\ (-n_y + i n_x)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} - i n_z \sin\frac{\theta}{2} \end{pmatrix}.$$

SO(3) :

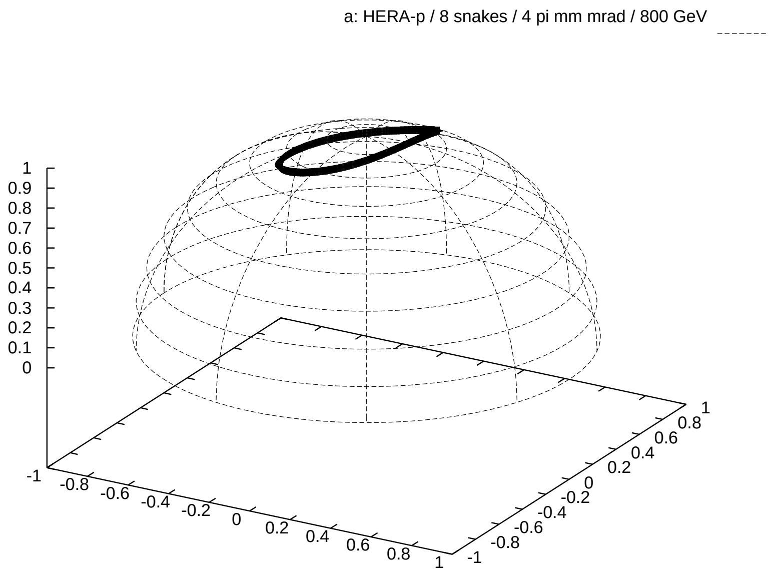
$$\begin{aligned} \mathbf{R}_{\hat{n}}(\theta) =& \mathbf{I}\cos\theta + \begin{pmatrix} 0 & n_z & -n_y \\ -n_z & 0 & n_x \\ n_y & -n_x & 0 \end{pmatrix} \sin\theta \\ &+ \begin{pmatrix} n_x^2 & n_x n_y & n_x n_z \\ n_x n_y & n_y^2 & n_y n_z \\ n_x n_z & n_y n_z & n_z^2 \end{pmatrix} (1 - \cos\theta). \end{aligned}$$

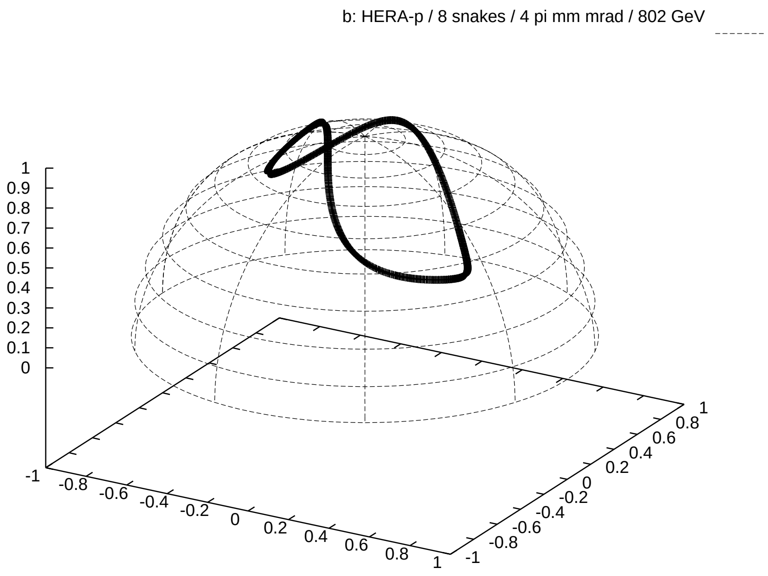
Invariant Spin Field



- For the closed orbit: $\vec{n}_0(s) = \vec{n}_0(s+L)$, with $\vec{q}_0(s) = \vec{q}_0(s+L)$ and $\vec{P}_0(s) = \vec{P}_0(s+L)$.
- For other locations in phase space: $\vec{n}(\vec{q}, \vec{P}, s) = \vec{n}(\vec{q}, \vec{P}, s+L)$, even though in general $q(s+L) \neq q(s)$ and $P(s+L) \neq P(s)$.

HERA-p: Invariant Spin Field





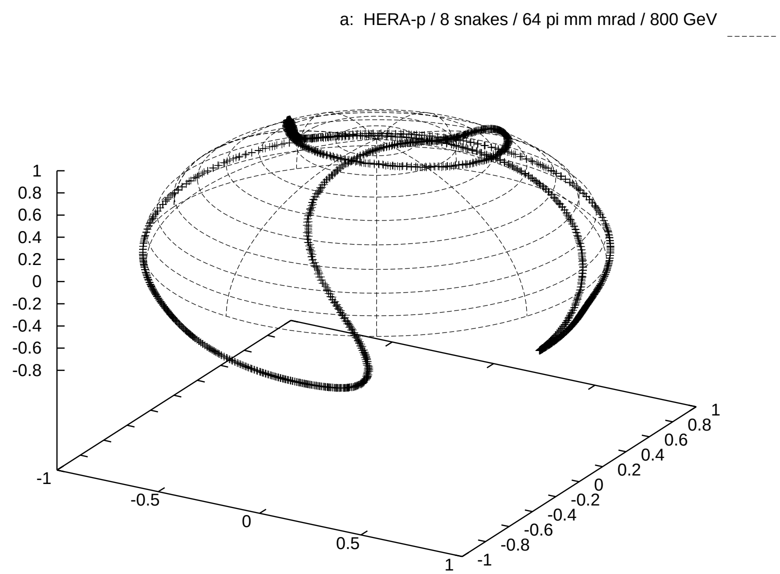
$\hat{n}$-vector at 1σ and 800 GeV

$\hat{n}$-vector at 1σ and 802 GeV

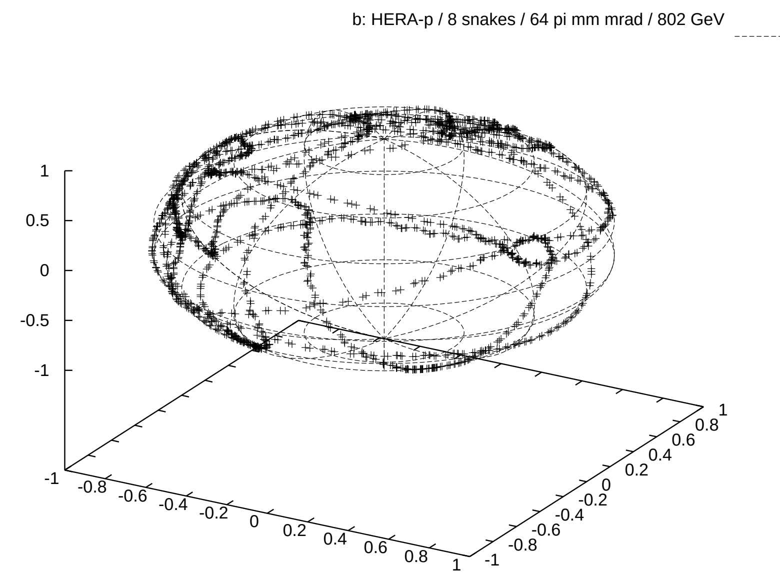
- Simulation with only vertical betatron motion.
- 802 GeV is closer to a resonance spin resonance than 800 GeV.

Des Barber et al.

HERA-p: Invariant Spin Field



$\hat{n}$-vector at 4σ and 800 GeV

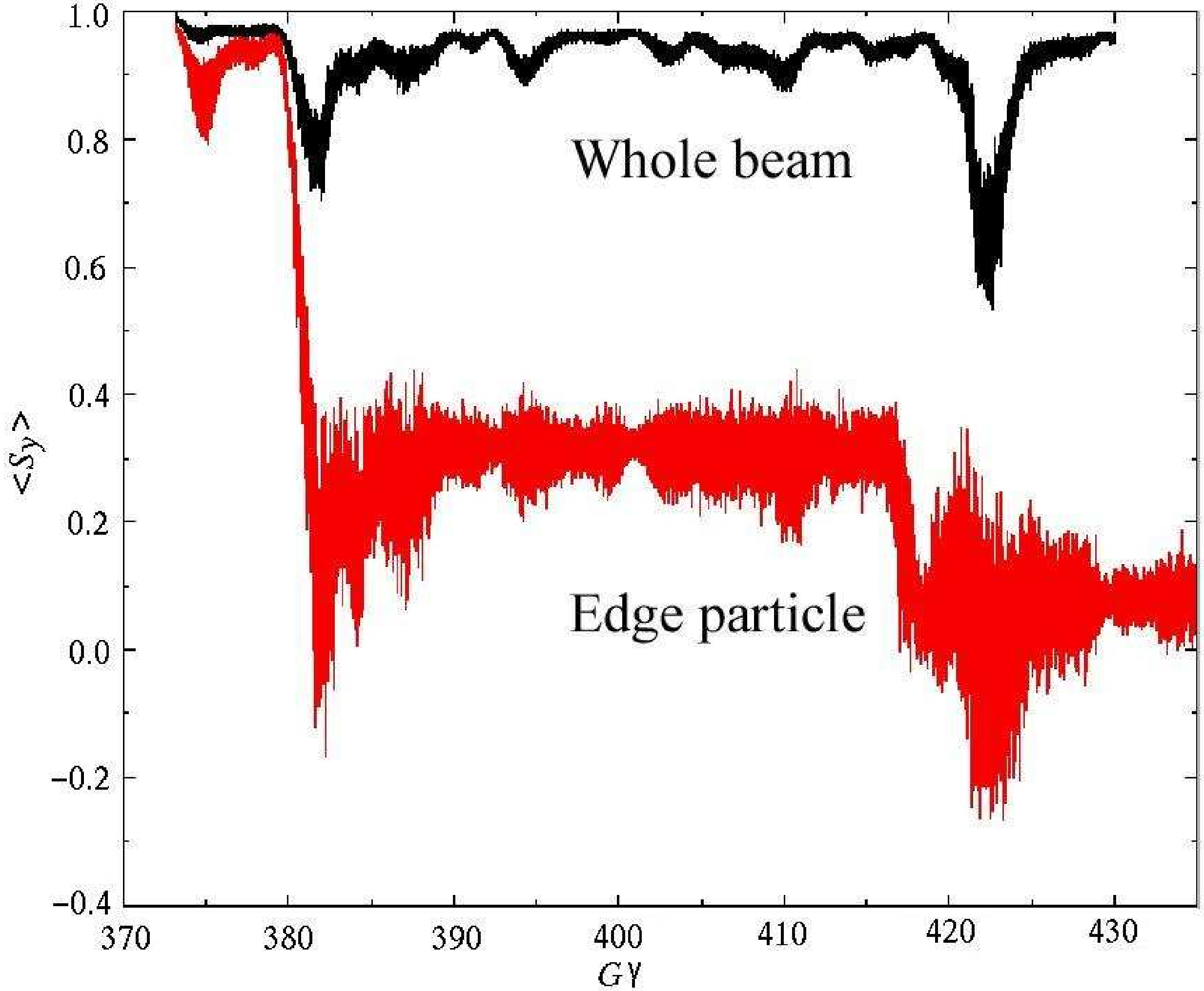


$\hat{n}$-vector at 4σ and 802 GeV

- Larger amplitude oscillations have a larger tune shift due to nonlinear elements.
- 802 GeV is closer to a resonance spin resonance than 800 GeV.

Des Barber et al.

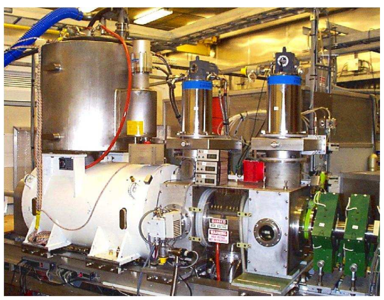
Spin Tracking in RHIC

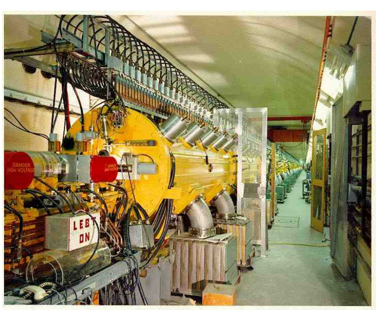


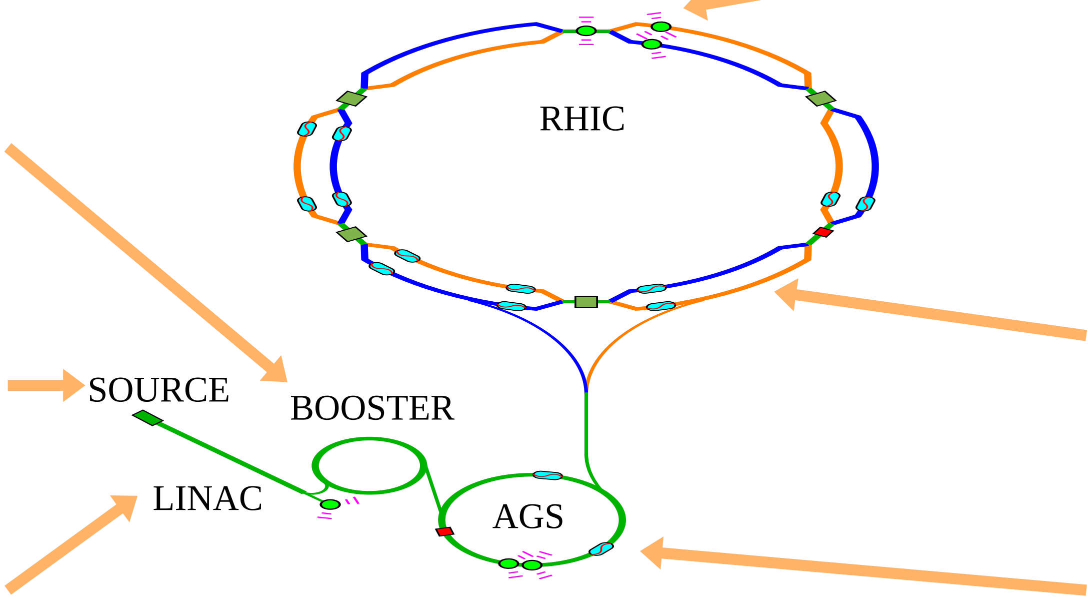
Particles with larger amplitude betatron oscillations may experience more precession away from the stable spin direction of the center of the beam

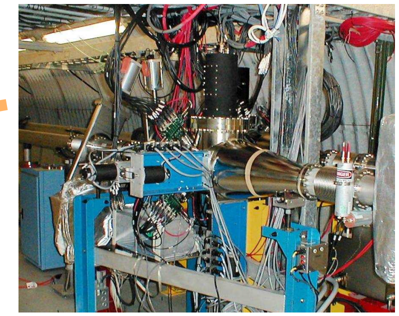
(Alfredo Luccio)

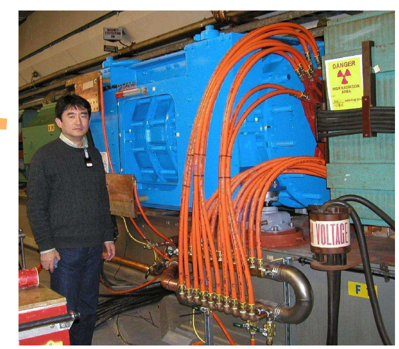
Accelerators with Polarized Protons





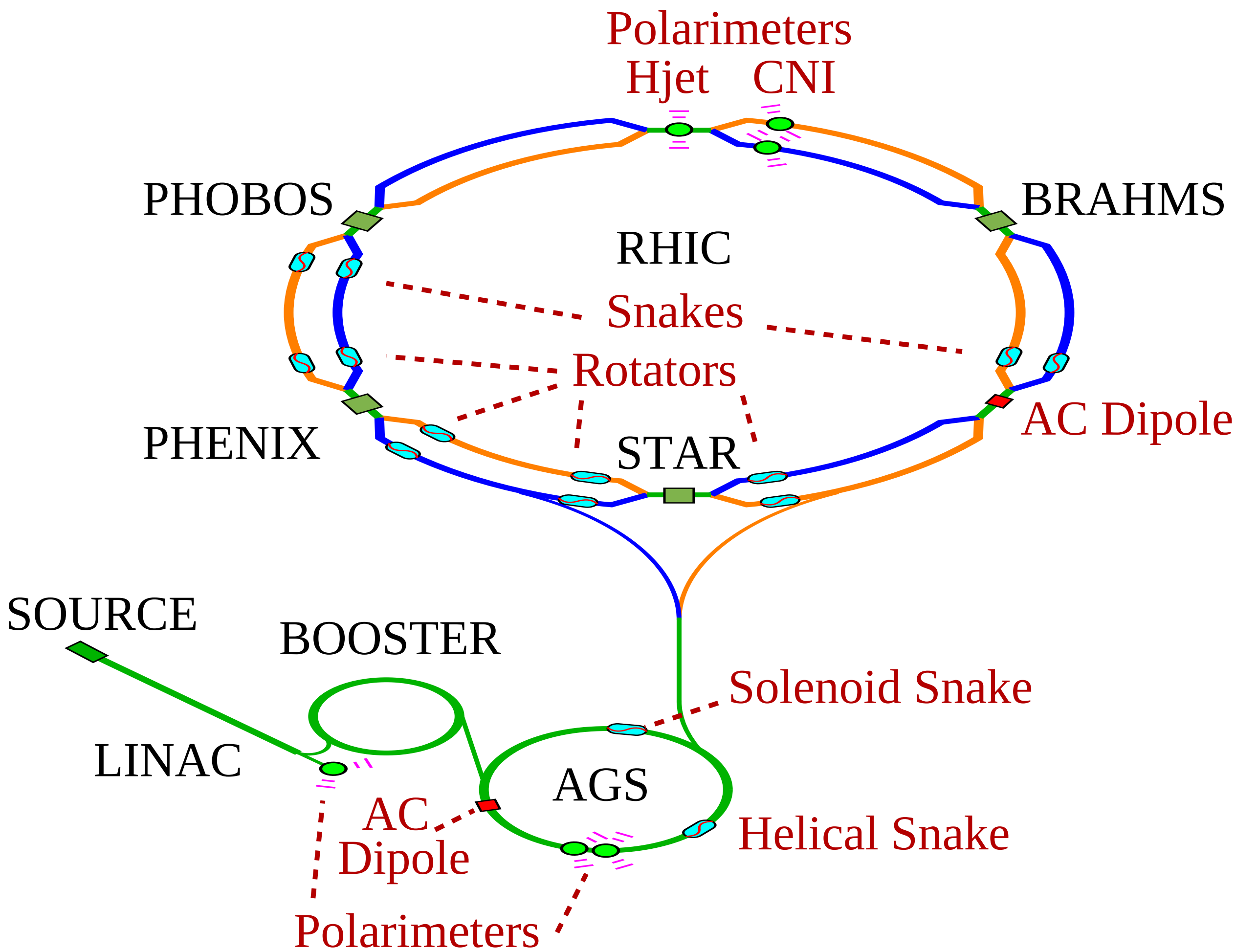






LINAC: Linear Accelerator
AGS: Alternating Gradient Synchrotron
RHIC: Relativistic Heavy Ion Collider

Accelerator Complex for Protons



High Intensity Polarized H⁻ Source

KEK OPPIS*
upgraded at TRIUMF

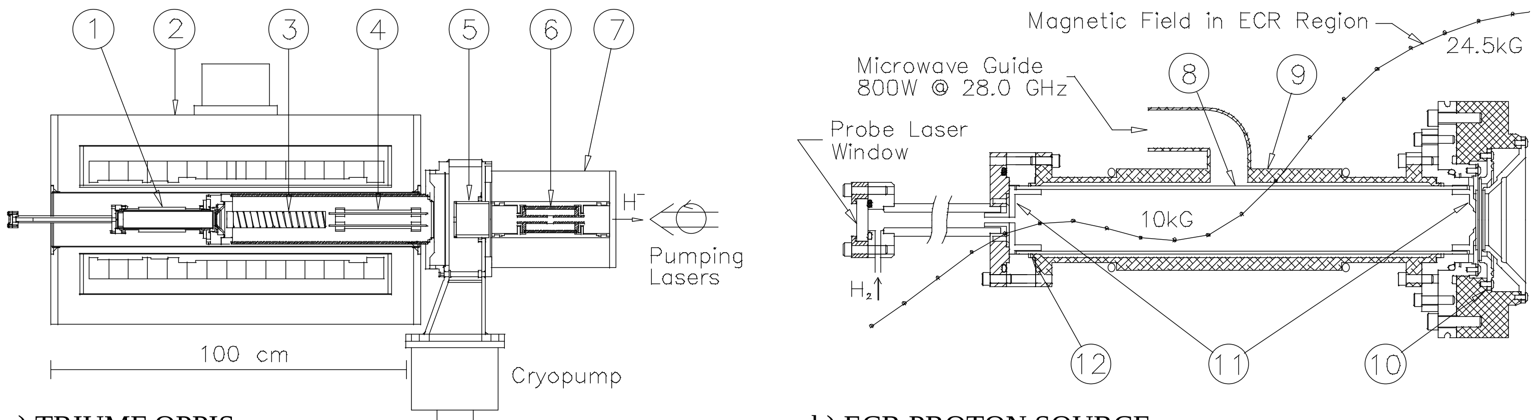
$70 \rightarrow 80\%$ Polarization

15×10^{11} protons/pulse
at source

6×10^{11} protons/pulse
at end of LINAC

*Optically Pumped Polarized Ion Source

Optically Pumped Polarized Ion Source



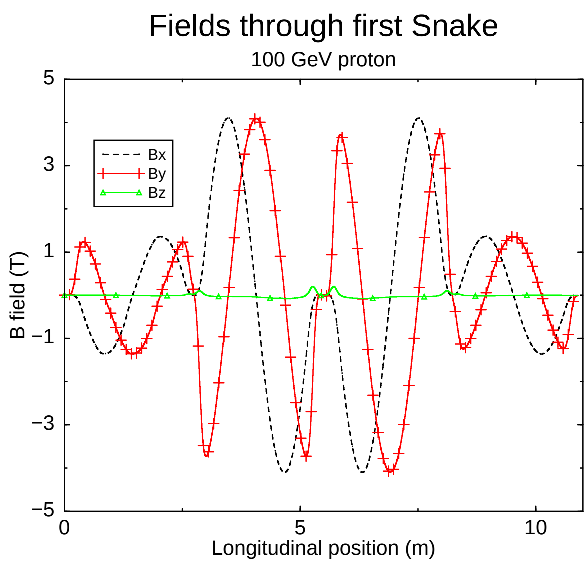
a) TRIUMF OPPIS b) ECR PROTON SOURCE

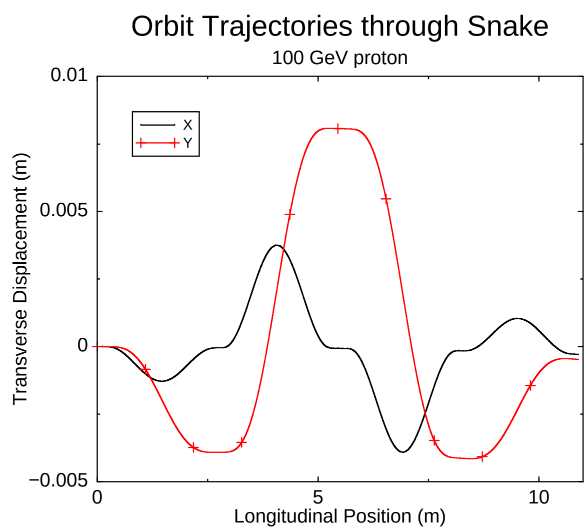
Fig. 1. 1) ECR Proton Source, 2) Superconducting Solenoid, 3) Optically-Pumped Rb Cell, 4) Deflection Plates, 5) Sona Transition Region, 6) Ionizer Cell, 7) Ionizer Solenoid, 8) Quartz Tube, 9) ECR Cavity, 10) Three Grid Extraction System, 11) Boron-Nitride End Cups, 12) Indium Seals.

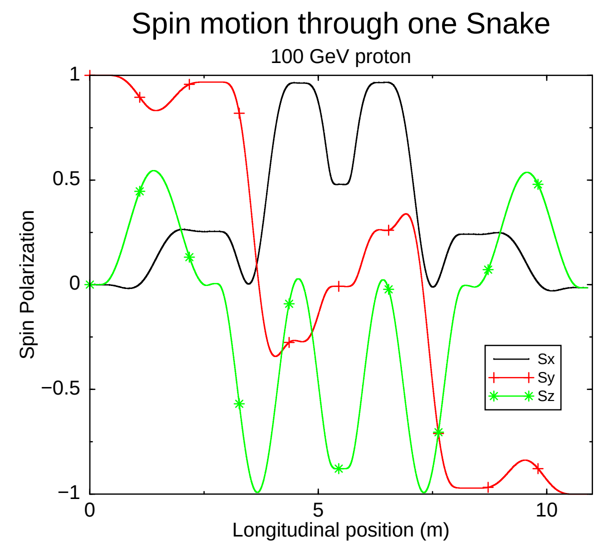
Generate protons → Capture polarized electron from optically pumped alkali atom → Transfer of electron-spin polarization to nuclear-spin polarization (Sona transition) → Ionization

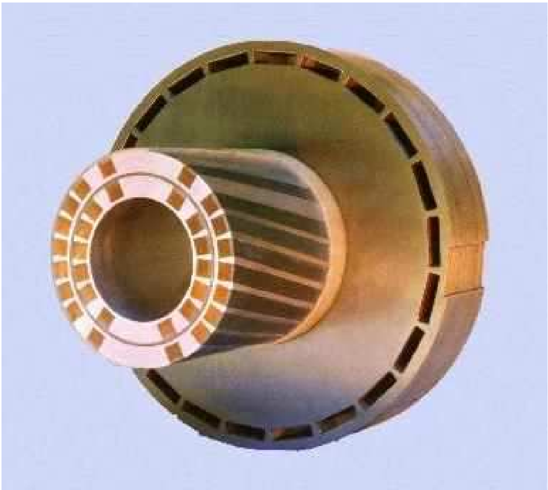
H^+ | H^0 $(e\uparrow)$ | H^0 $(p\uparrow)$ | H^- $(p\uparrow)$

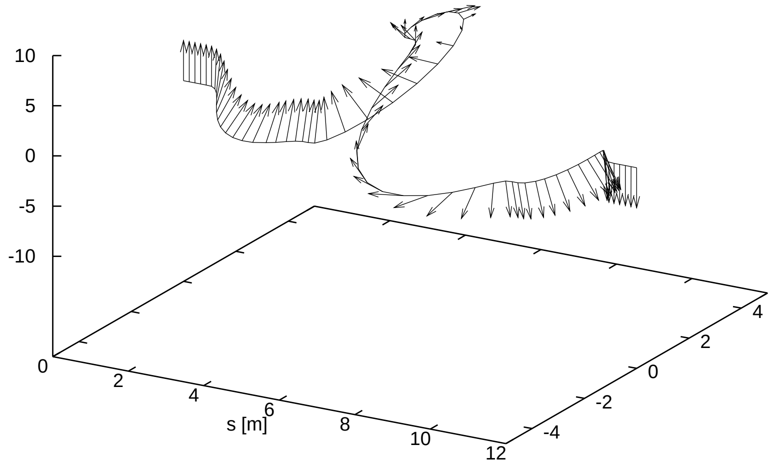
Trajectory and Spin through Snakes



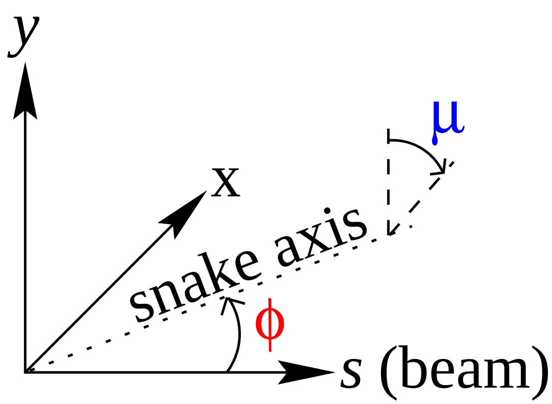






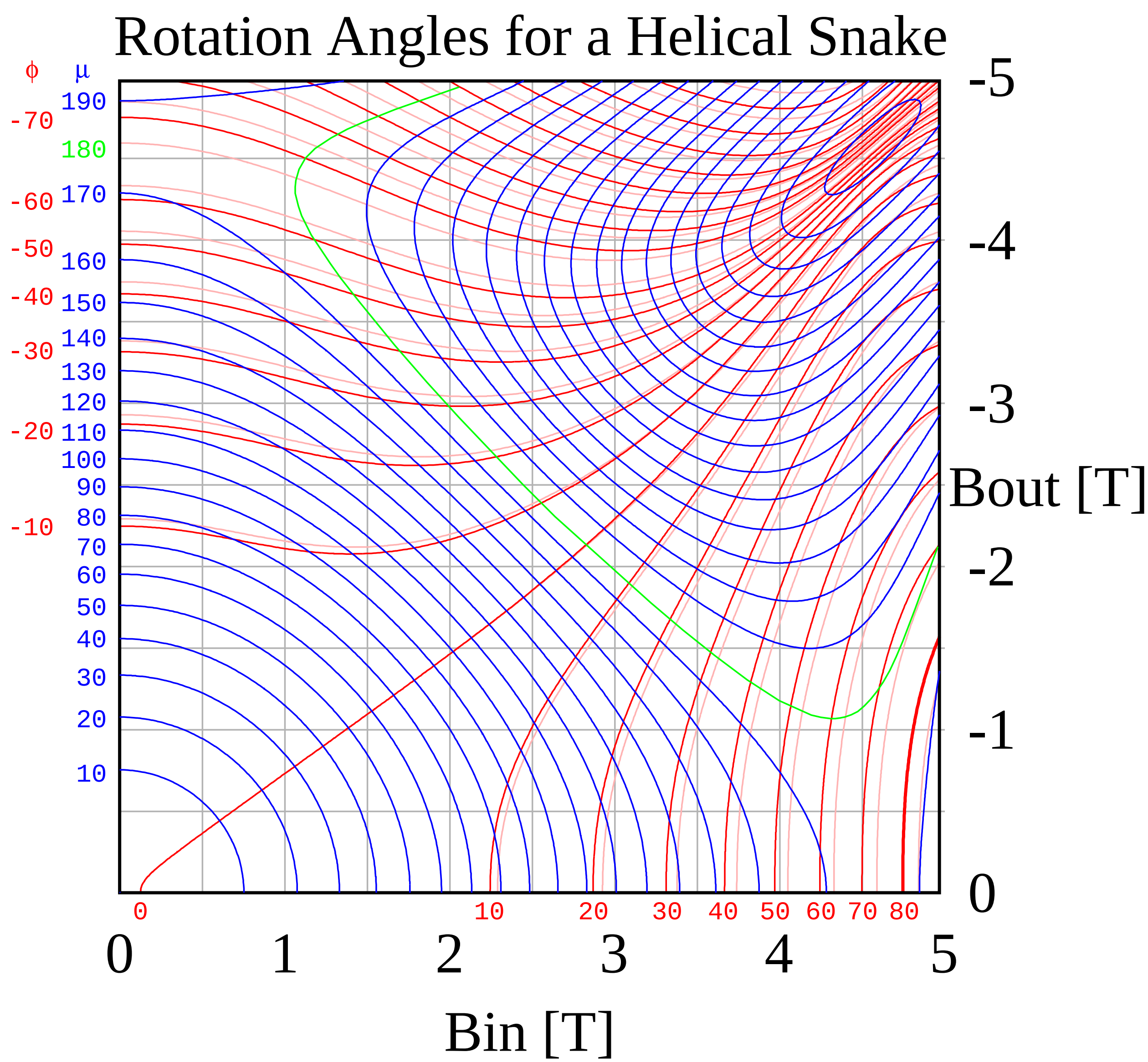


BROOKHAVEN NATIONAL LABORATORY

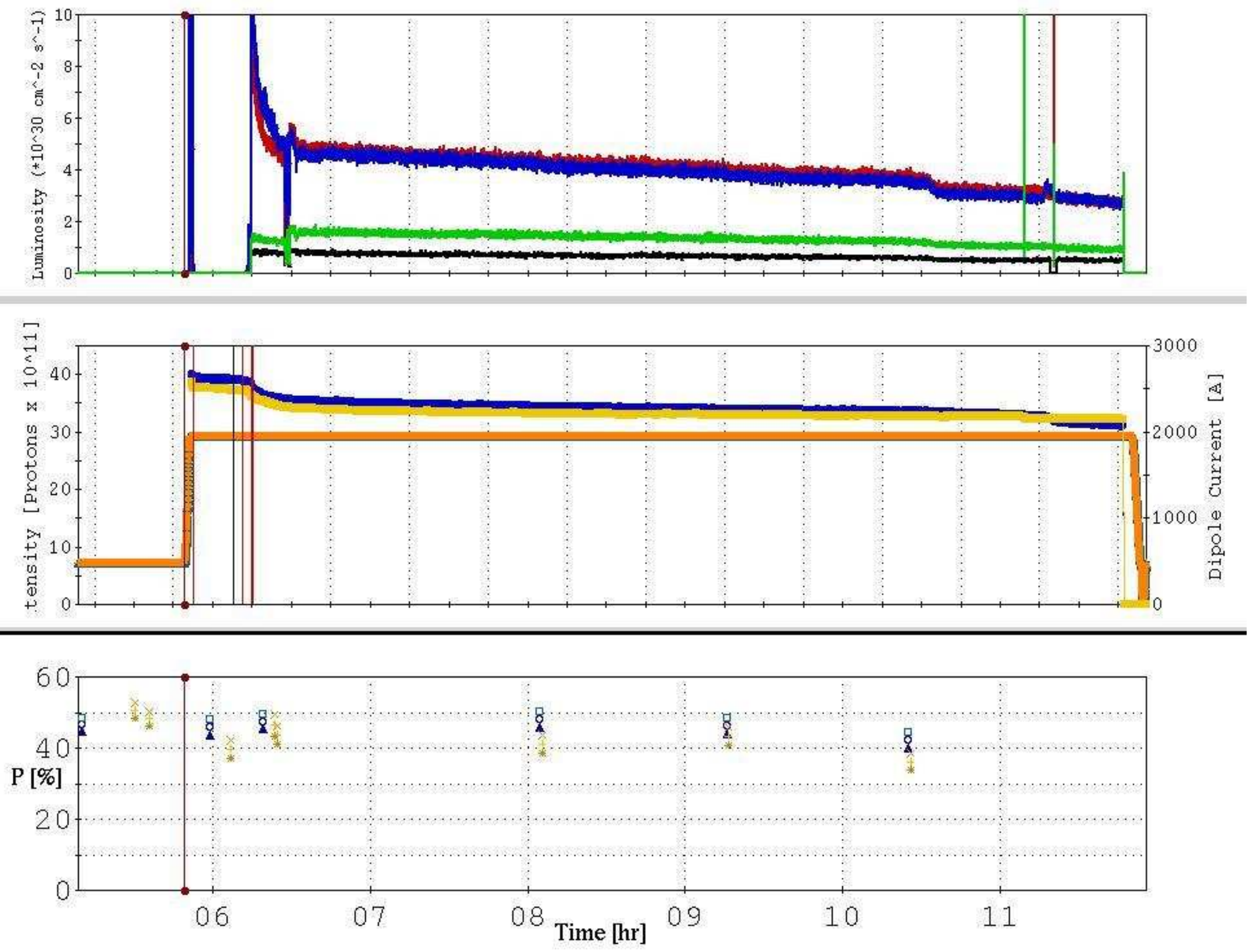


The rotation axis of the snake is ϕ, and μ is the rotation angle.

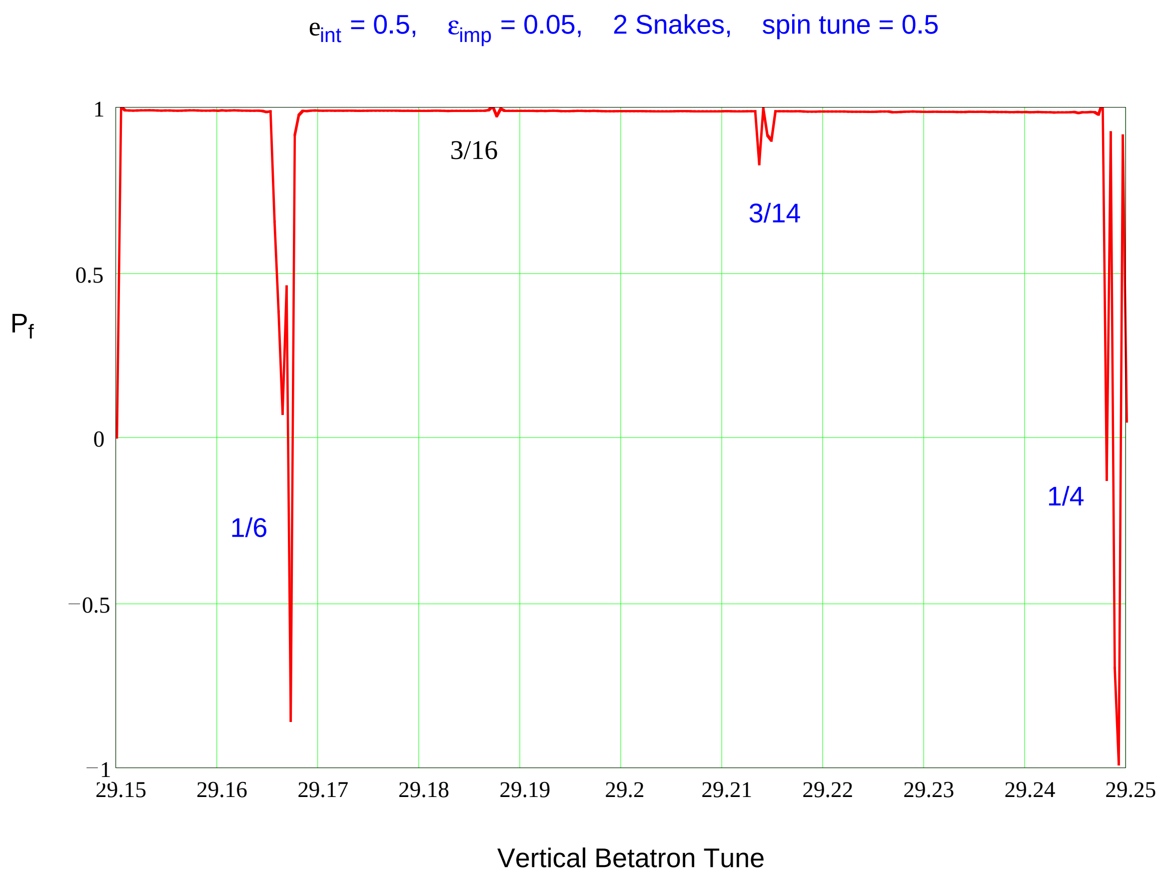
Note that the ϕ contours shift slightly from injection (red) at 25 GeV to storage (pink) at 250 GeV.



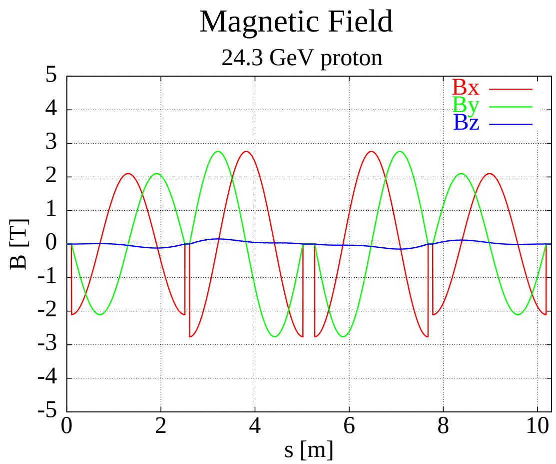
RHIC Beam Polarization

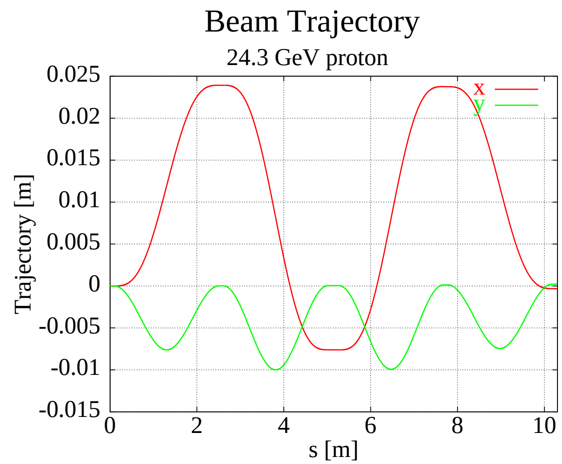


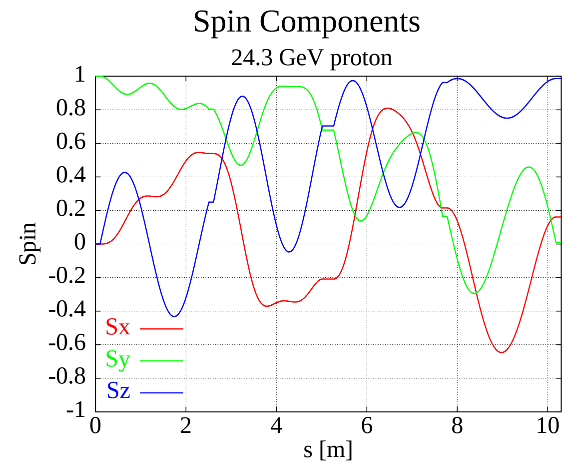
Snake Resonances

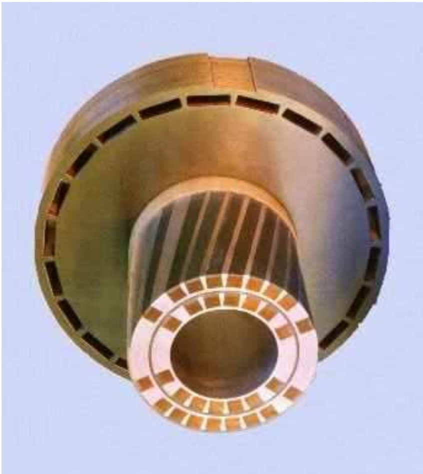


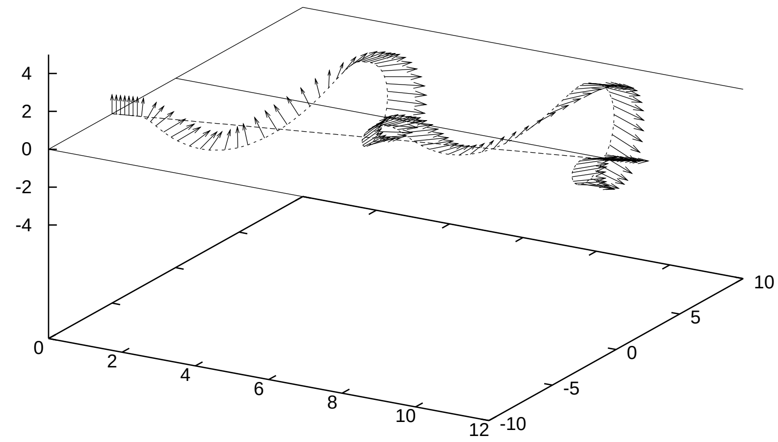
Helical Spin Rotators



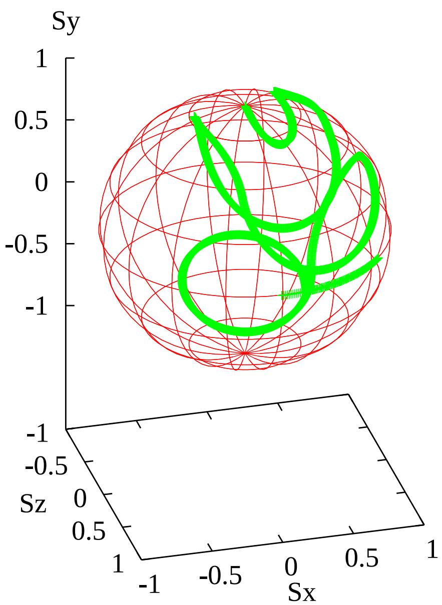




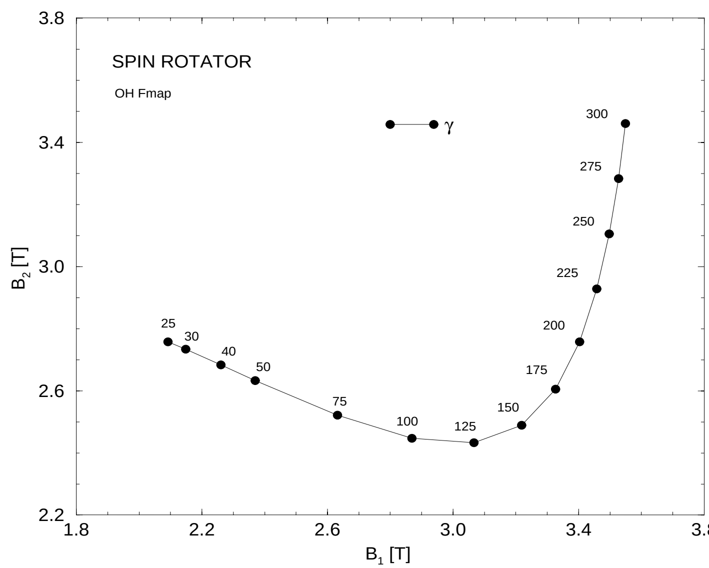


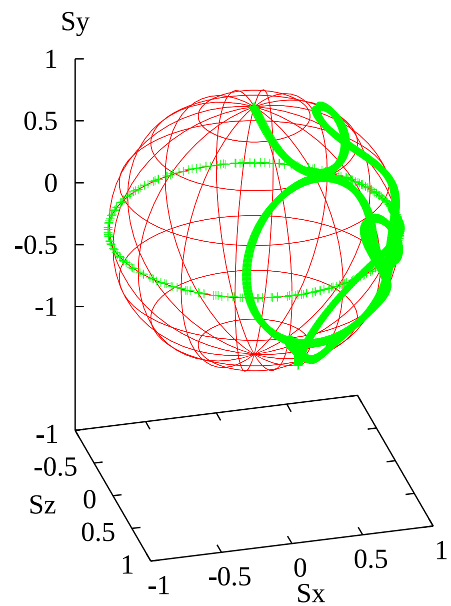


Compensation for D0-DX Bends

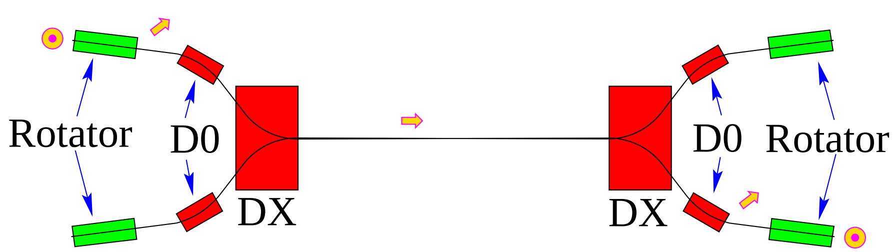


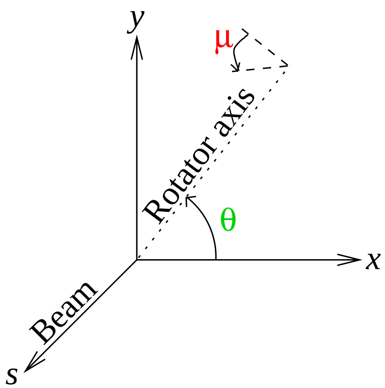
$E = 25$ GeV



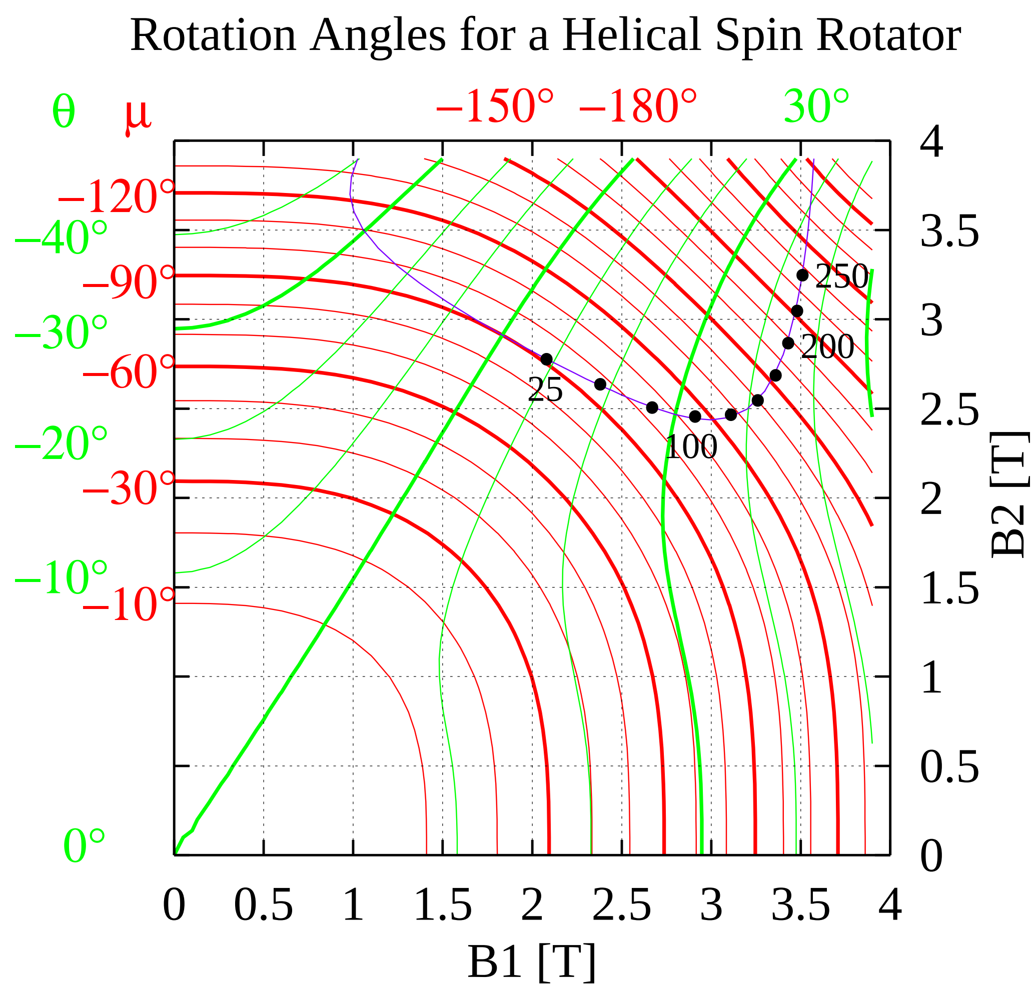


$E = 250$ GeV



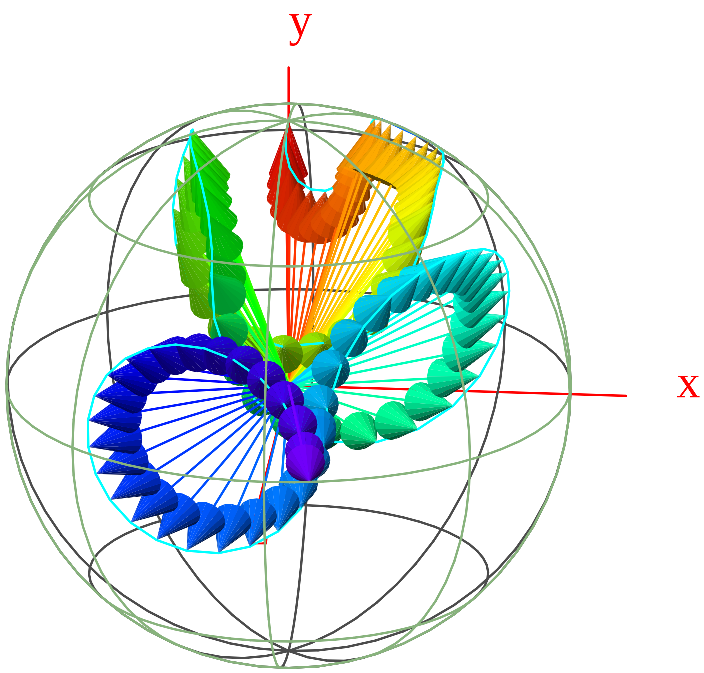


The rotation axis of the spin rotator is in the x-y plane at an angle θ from the vertical. The spin is rotated by the angle μ around the rotation axis.

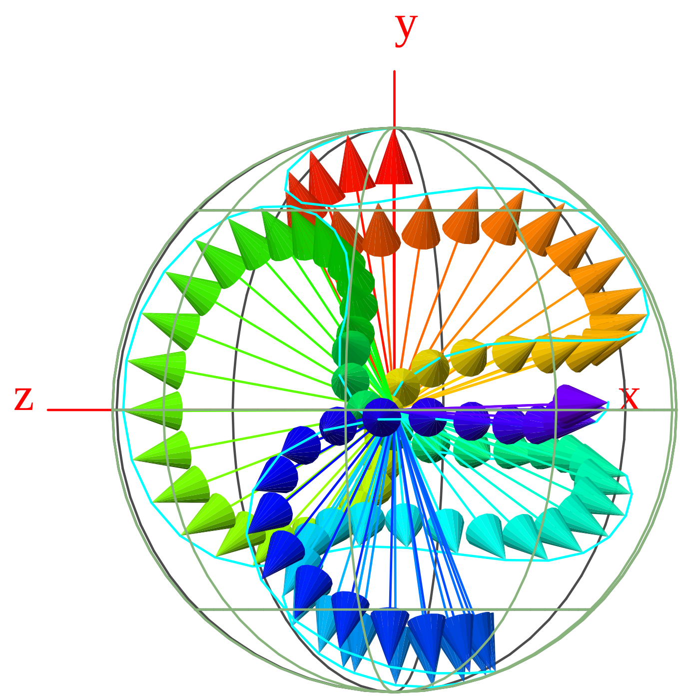


Note: Purple contour for rotation into horizontal plane.
Black dots show settings for RHIC energies in increments of 25 GeV from 25 to 250 GeV.

Rotator Spin Precession



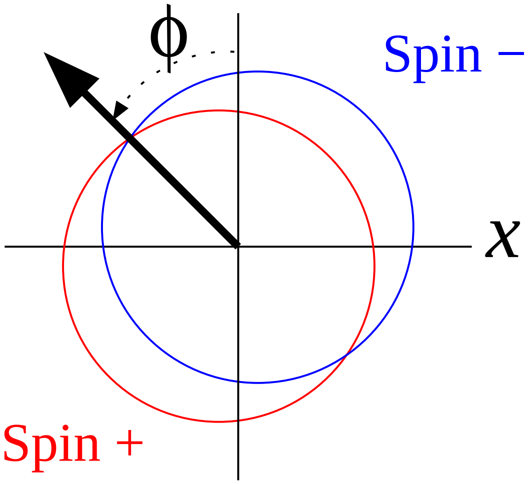
Rotator's spin vector at injection energy



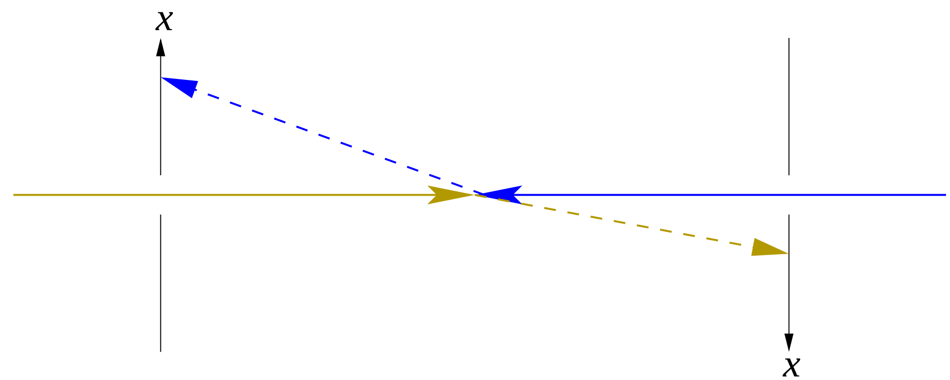
Rotator's spin vector at 250 GeV

BROOKHAVEN NATIONAL LABORATORY

Orientation of PHENIX Polarimeters



"Left−Right" Asymmetry
(Tilted at 45°)

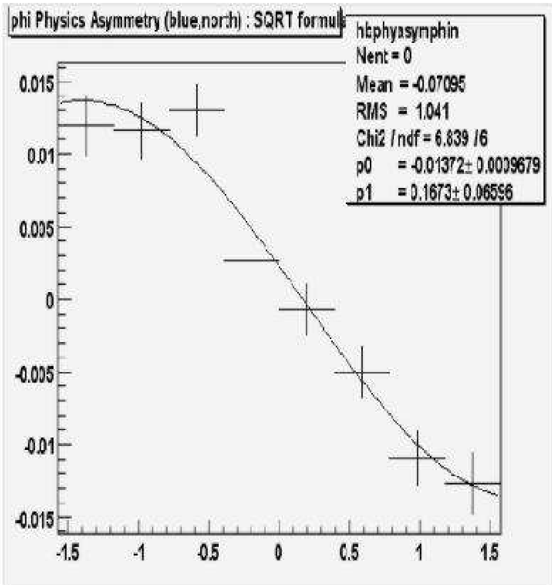


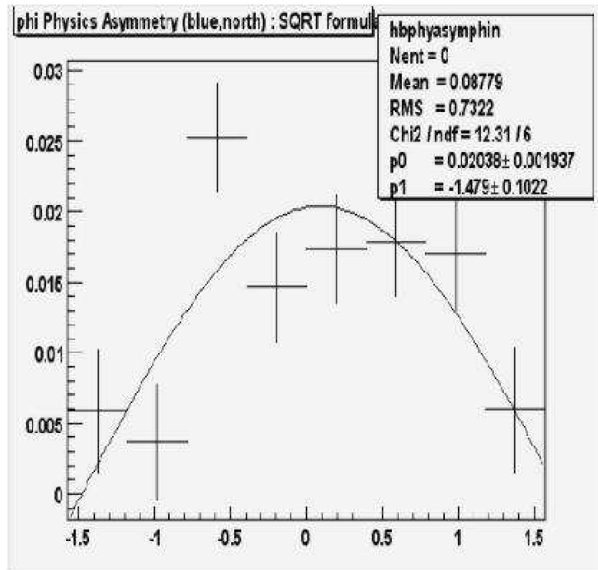
Schematic layout of PHENIX polarimeters
Yellow from left. Blue from right.

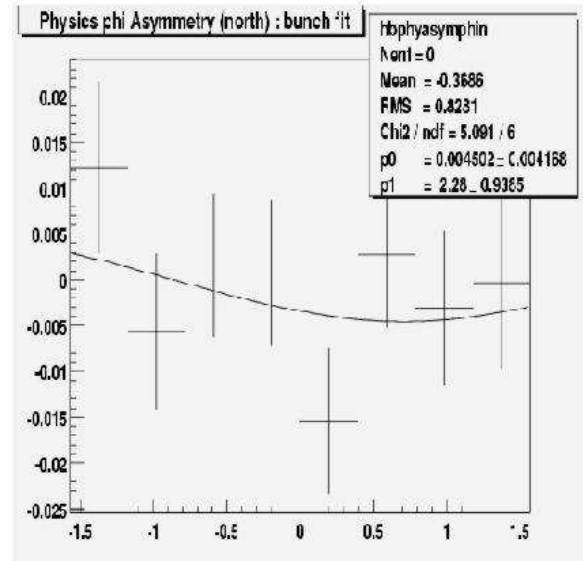
The PHENIX Local Polarimeter measures an asymmetry in small angle scattered neutrons which is proportional to transverse polarization.

$$A_{\mathrm{LR}} = \frac{\sqrt{L^+R^-} - \sqrt{L^-R^+}}{\sqrt{L^+R^-} + \sqrt{L^-R^+}} \propto P_y$$

Tale of the Blue Ring







Vertical polarization with rotators off.

Spin is down.

Rotators on

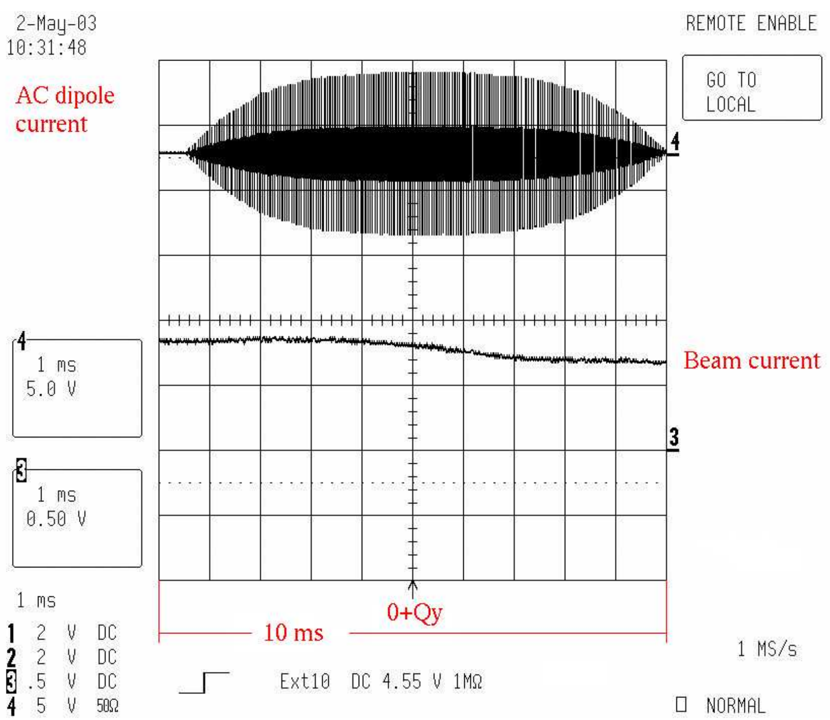
Spin is radially inwards!

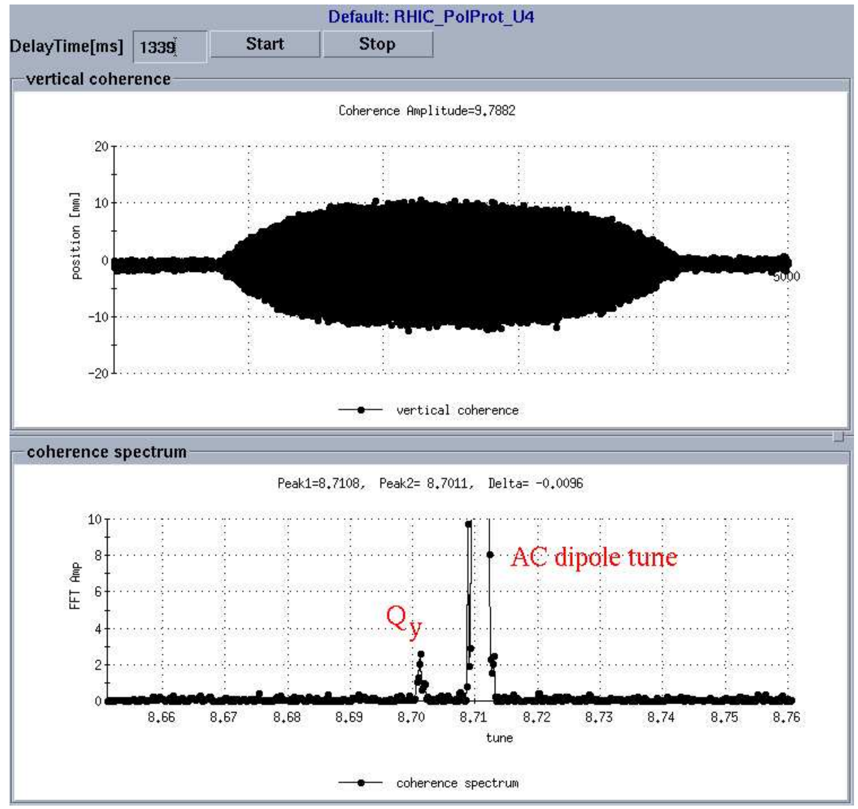
OOPS!

Reverse all rotator power supplies and try again.

YES!

$\mathcal{L}$ AC Dipole pulse at $G\gamma = 0 + Q_y$ $\mathcal{J}$





Top: AC dipole pulse amplitude
(current)

Bottom: Beam current.
(Just scrapes the beam pipe.)

Top: Beam coherence

Bottom: Tune spectrum

Properties of synchrotron radiation

- Radiated power:

$$P_\gamma = \frac{2}{3} r_e mc^3 \frac{\gamma^4 \beta^4}{\rho^2}, \qquad r_e = \frac{e^2}{4\pi\epsilon_0\, mc^2}.$$

Radiation in forward direction with opening angle $\propto \gamma^{-1}$

- Energy loss per turn:

$$U_\gamma = \oint \frac{P_\gamma}{c} ds$$

- Critical energy: half the power is radiated by photons less than the critical energy, and the other half, above.

$$u_c = \hbar\omega_c = \frac{3\hbar c}{2\rho}\gamma^3$$

- Number of photons per second:

$$N_\gamma = \int_0^{U_{\max}} n_\gamma(u_\gamma)\, du_\gamma = \frac{5}{2\sqrt{3}} \frac{\alpha c}{\rho}\gamma$$

here: $\alpha = 1/137)$

- Number of photons per radian:

$$N_r = \frac{5\alpha}{2\sqrt{3}}\gamma$$

- Average photon energy and 2nd moment:

$$\langle u_\gamma \rangle = \frac{1}{N_\gamma}\int_0^{U_{\max}} u\, n_\gamma(u)\, du = \frac{8}{15\sqrt{3}} u_c \simeq 0.32 u_c$$

$$\langle u_\gamma^2 \rangle = \frac{1}{N_\gamma}\int_0^{U_{\max}} u^2 n_\gamma(u)\, du = \frac{11}{27\sqrt{3}} u_c^2 \simeq 0.41 u_c^2$$

- Energy spread: $$\sigma_u = \sqrt{\frac{C_q}{J_s \rho}} \gamma^2 mc^2$$

with $C_q = 3.8 \times 10^{-8}$ m and $J_s \sim 2 + \mathcal{D}$.

Ring	Energy [GeV]	σ_u [MeV]
CESR	5.5	3
HERAe	27.5	3
LEP	45	30
LEP	60	53
LEP	100	150

Remember: Integer resonances separated by only 440 MeV.

The polarization in LEP dropped down to nothing just above 60 GeV.

Longitudinal Synchrotron Oscillations

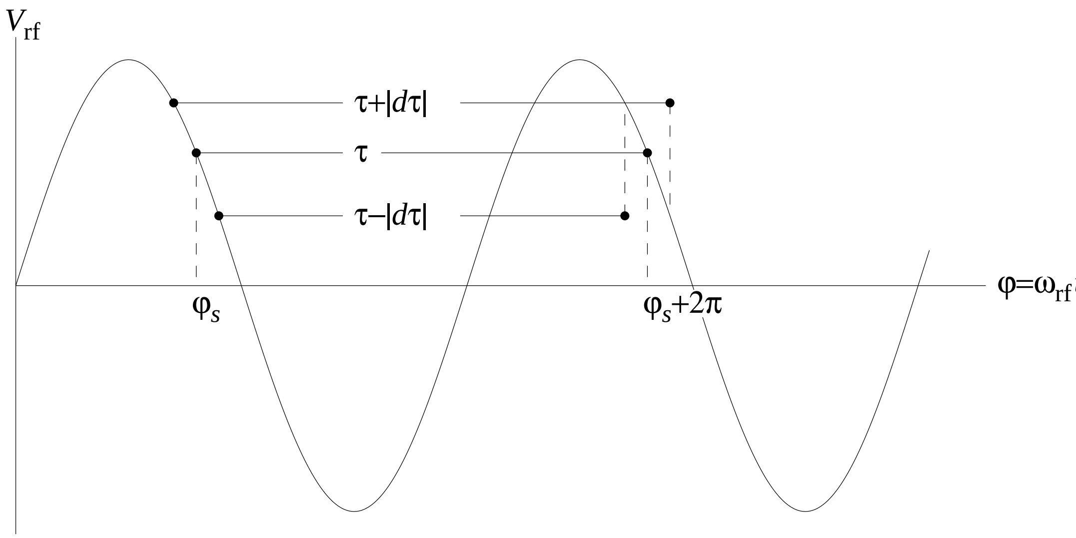
$$\omega_{\rm rf} = h\omega_{\rm rev}$$

$$W = -\frac{U - U_s}{\omega_{\rm rf}}$$

$$\frac{dW}{dt} = \frac{qV}{2\pi h}(\sin\phi_s - \sin\phi)$$

$$\frac{d\varphi}{dt} \simeq \frac{\omega_{\rm rf}^2 \eta_{\rm ph}}{\beta^2 U_s} W$$

$$\frac{d\omega_{\rm rev}}{\omega_{\rm rev}} = \frac{d\beta}{\beta} - \frac{dL}{L} = \eta_{\rm ph}\frac{dp}{p}$$



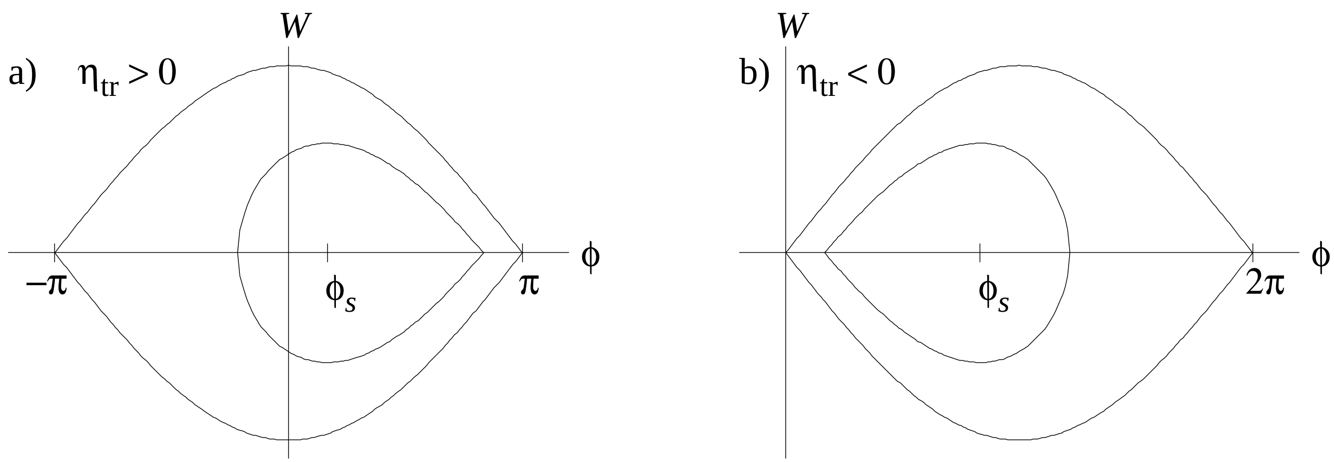
$\eta_{\rm ph} < 0$ above transition energy.

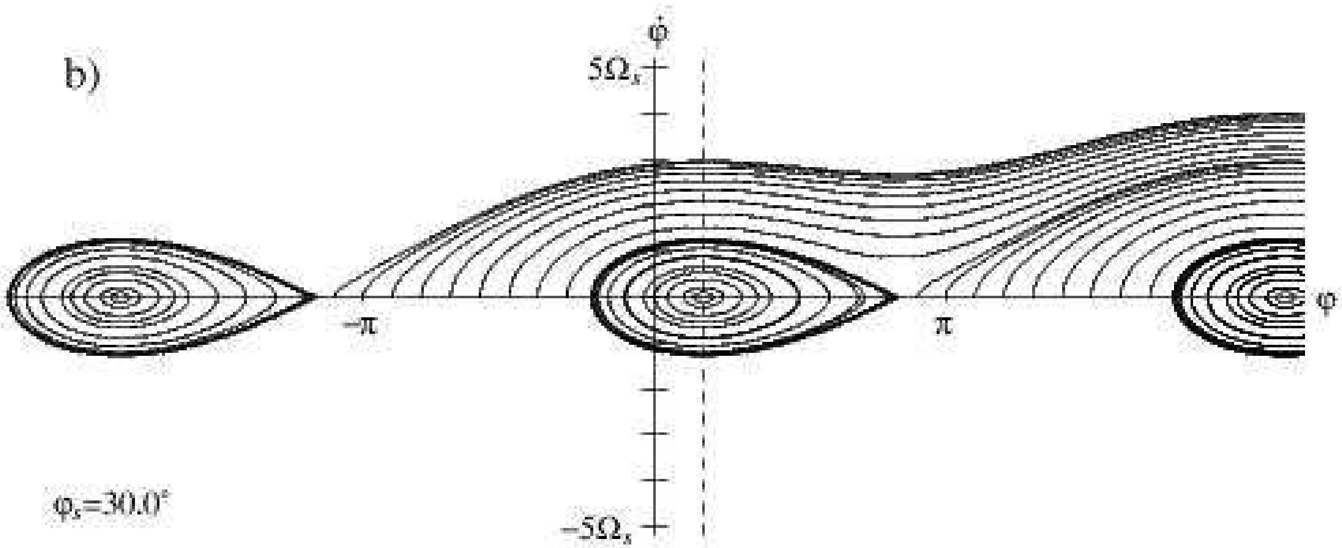
Add in synchrotron oscillations to resonance condition:

$$\nu_{\rm spin} = N + N_{\rm v}Q_{\rm v} + N_{\rm h}Q_{\rm h} + N_{\rm sy}Q_{\rm sy}$$

Longitudinal Phase Space

Canonical cordinate: φ and conjugate momentum: W





Spin-flip Transition Rates

In a homogenious magnetic field the transition rates are

$$W_{\uparrow\downarrow} = \frac{5\sqrt{3}}{16}\,\frac{e^2\gamma^5\hbar}{4\pi\epsilon_0 m_{\mathrm{e}} c^2|\rho|^3}\left(1 + \frac{8}{5\sqrt{3}}\right)$$

$$W_{\uparrow\downarrow} = \frac{5\sqrt{3}}{16}\,\frac{e^2\gamma^5\hbar}{4\pi\epsilon_0 m_{\mathrm{e}} c^2|\rho|^3}\left(1 - \frac{8}{5\sqrt{3}}\right).$$

Evaluating the equilibrium polarization have (Sokolov Ternov)

$$P_{\mathrm{ST}} = \frac{W_{\uparrow\downarrow} - W_{\downarrow\uparrow}}{W_{\uparrow\downarrow} + W_{\downarrow\uparrow}} = \frac{8}{5\sqrt{3}} = 0.9238.$$

An unpolarized beam polarizes:

$$P(t) = P_{\mathrm{ST}}\left[1 - \exp(-t/\tau_{\mathrm{ST}})\right],$$

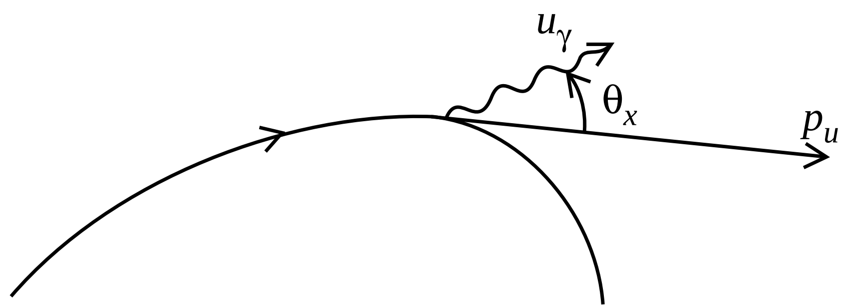
where the polarization rate is given by

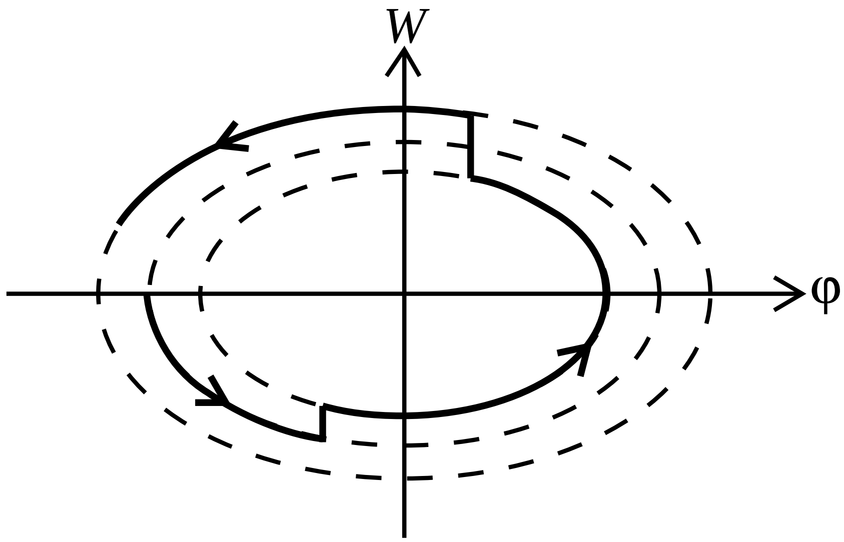
$$\tau_{\mathrm{ST}}^{-1} = \frac{5\sqrt{3}}{8}\,\frac{e^2\gamma^5\hbar}{4\pi\epsilon_0 m_{\mathrm{e}}^2 c^2}\,\frac{1}{L}\oint\frac{ds}{|\rho|^3}.$$

Typical Sokolov-Ternov Rates

Ring	Particle	Energy [GeV]	N_γ [/turn]	ΔU [loss/turn]	$\tau_{\rm ST}$	$\frac{W_{\uparrow\downarrow}}{f_{\rm rev} N_\gamma}$
CESR	$e^\pm$	5.5	700	-1 MeV	167 min	1×10^{-13}
HERAe	$e^\pm$	27.5	3600	-83 MeV	23 min	1×10^{-12}
LEP	$e^\pm$	45	5800	-120 MeV	300 min	2×10^{-13}
LEP	$e^\pm$	60	7800	-380 MeV	81 min	8×10^{-13}
RHIC	p	100	7	-3 meV	3×10^{14} yr	6×10^{-29}
RHIC	p	250	18	-0.13 eV	3×10^{12} yr	2×10^{-27}
HERAp	p	920	65	-8.5 eV	1×10^{11} yr	3×10^{-26}
Tevatron	p	1000	70	-8.5 eV	2×10^{11} yr	2×10^{-26}
SSC	p	20000	1400	-0.12 MeV	7×10^{7} yr	3×10^{-23}

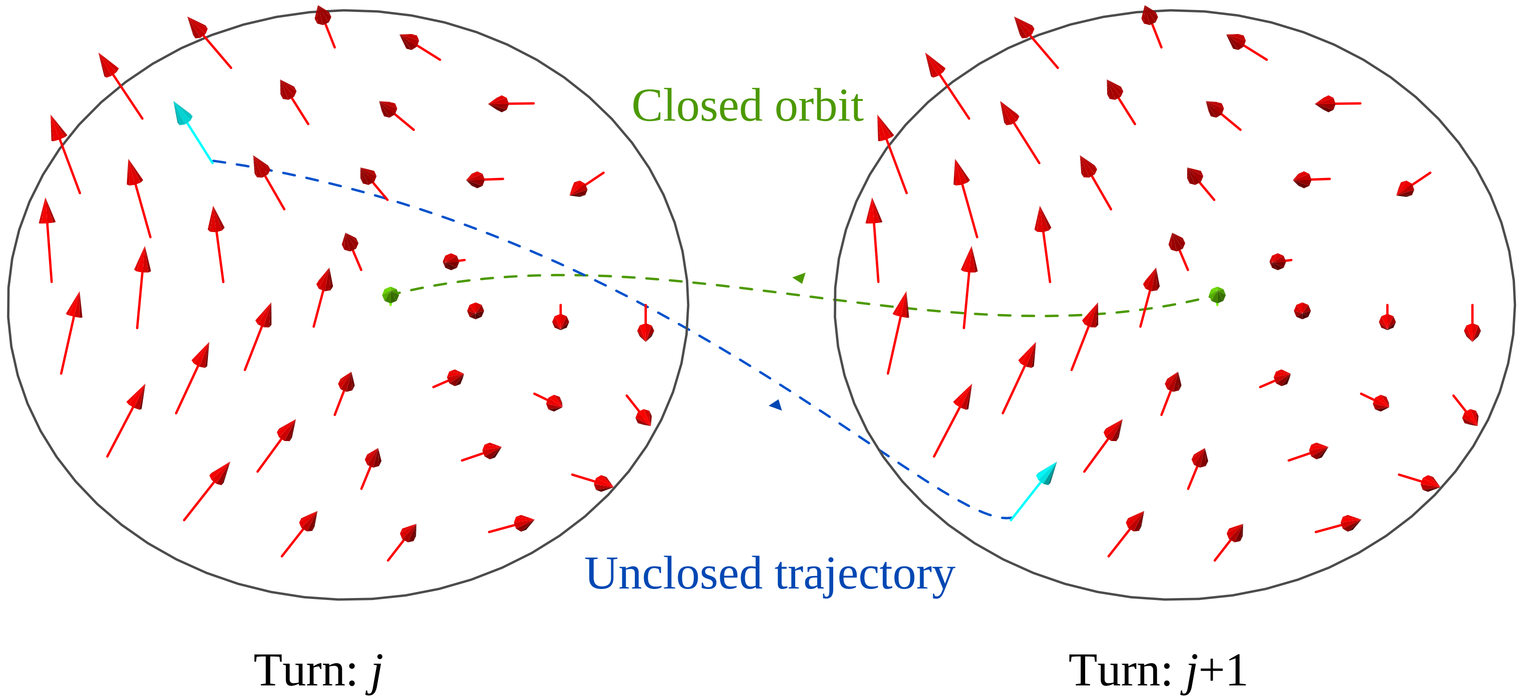
Quantum Fluctuations





In phase space quantum fluctuations cause instantaneous hops of momentum from one ellipse to another. (Hops in the Action.)

Invariant Spin Field



- For the closed orbit: $\vec{n}_0(s) = \vec{n}_0(s+L)$,
  with $\vec{q}_0(s) = \vec{q}_0(s+L)$ and $\vec{P}_0(s) = \vec{P}_0(s+L)$.
- For other locations in phase space: $\vec{n}(\vec{q}, \vec{P}, s) = \vec{n}(\vec{q}, \vec{P}, s+L)$,
  even though in general $q(s+L) \neq q(s)$ and $P(s+L) \neq P(s)$.

Equilibrium with Real Lattice

Derbenev–Kondratenko formula for equilibrium polarization:

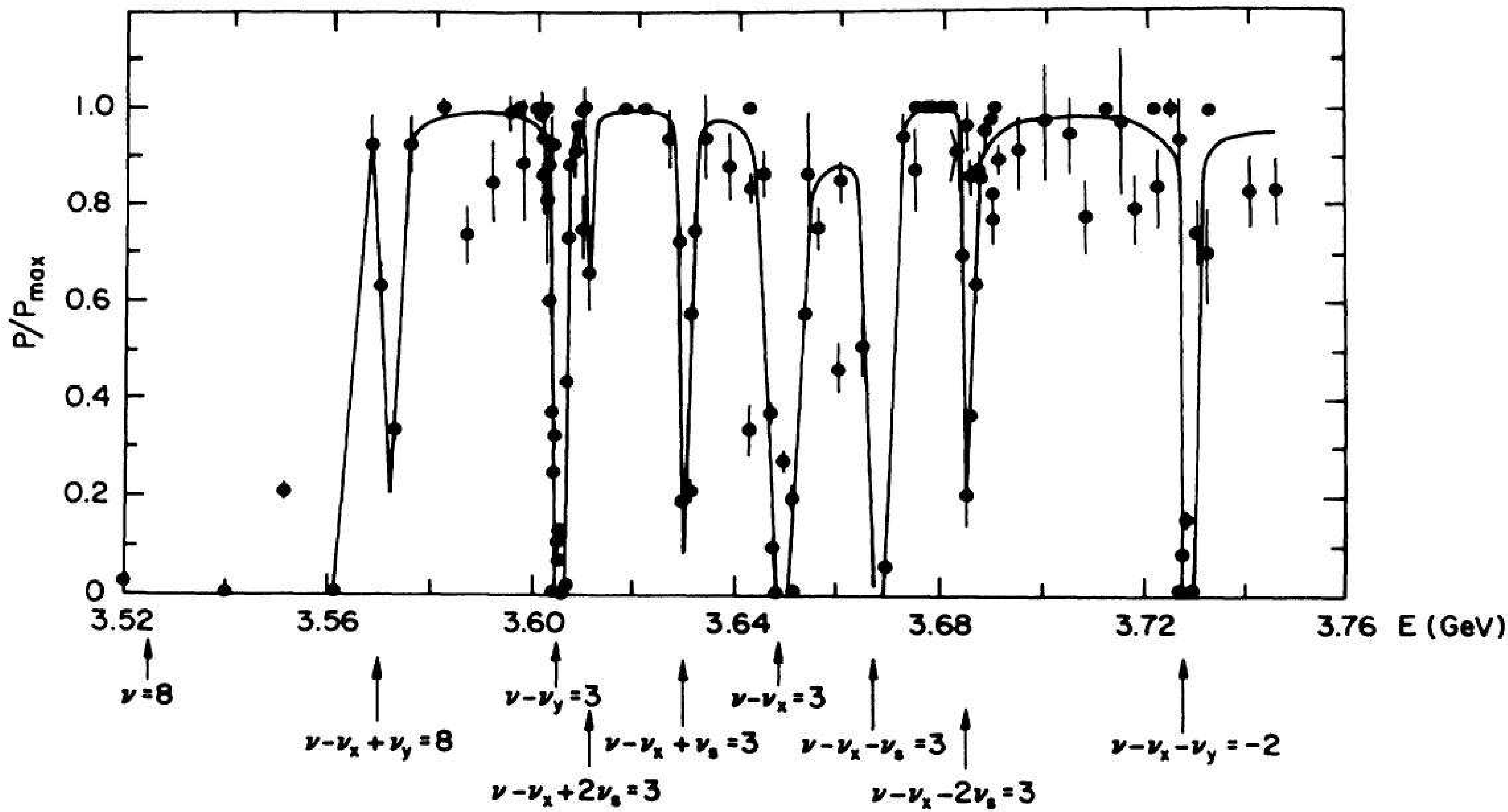
$$P_{\mathrm{DK}} = \frac{8}{5\sqrt{3}} \frac{\oint \left\langle \frac{1}{|\rho|^3} \hat{b} \cdot \left(\hat{n} - \frac{\partial \hat{n}}{\partial \delta}\right) \right\rangle_s ds}{\oint \left\langle \frac{1}{|\rho|^3} \left[1 - \frac{2}{9}(\hat{n}\cdot\hat{s})^2 + \frac{11}{18}\left(\frac{\partial \hat{n}}{\partial \delta}\right)^2\right] \right\rangle_s ds}$$

$$\frac{1}{\tau_{\mathrm{DK}}} = \frac{5\sqrt{3}}{8} \frac{r_e \gamma^5 \hbar}{m_e} \frac{1}{L} \oint \left\langle \frac{1}{|\rho|^3} \left[1 - \frac{2}{9}(\hat{n}\cdot\hat{s}) + \frac{11}{18}\left(\frac{\partial \hat{n}}{\partial \delta}\right)^2\right] \right\rangle_s ds$$

averaged over phase space at azimuth s.

$\delta = \Delta p/p$ is the fractional momentum deviation from design.
$\hat{n}$ is the invariant spin field.
$\hat{b} = \frac{\hat{s}\times\dot{\hat{s}}}{|\dot{\hat{s}}|}$ is the direction of magnetic field if $\vec{E} = 0$.
ρ is the cyclotron radius of the trajectory.
L is circumference of synchrotron.

Spin Resonances of SPEAR

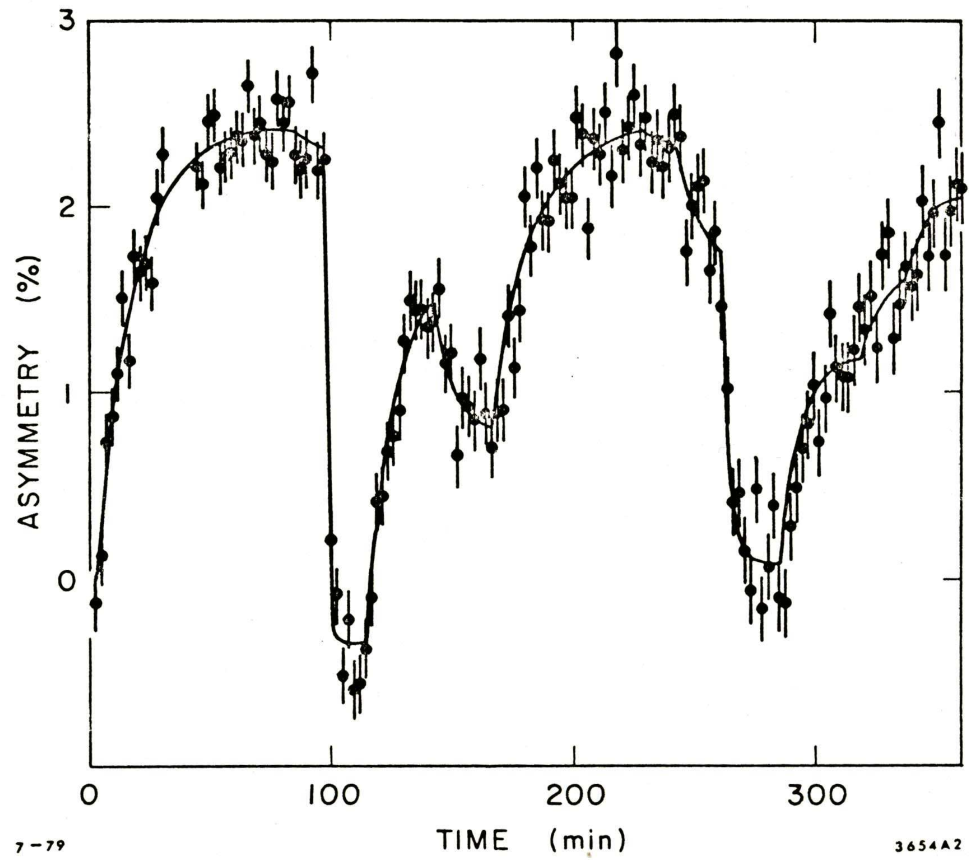


BROOKHAVEN NATIONAL LABORATORY

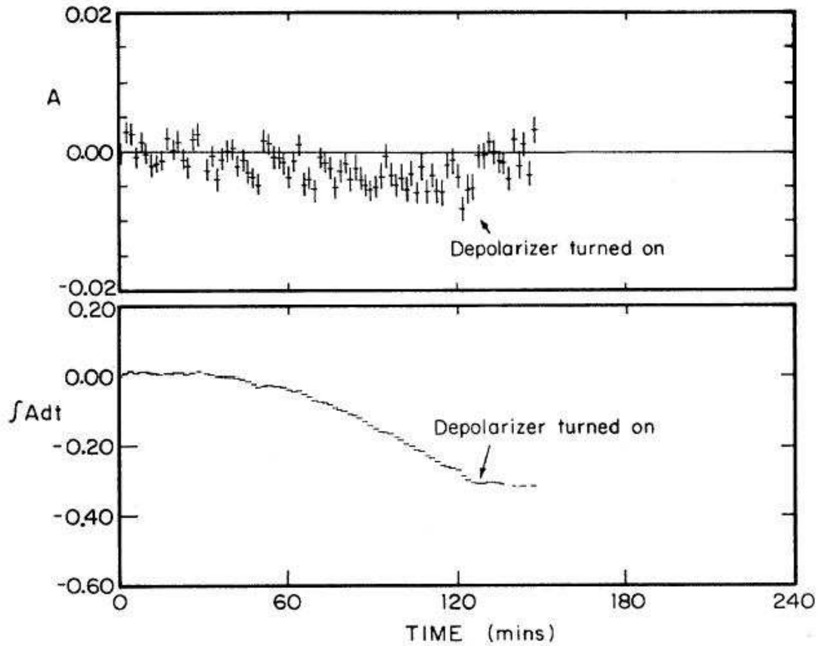
SPEAR Polarization Time

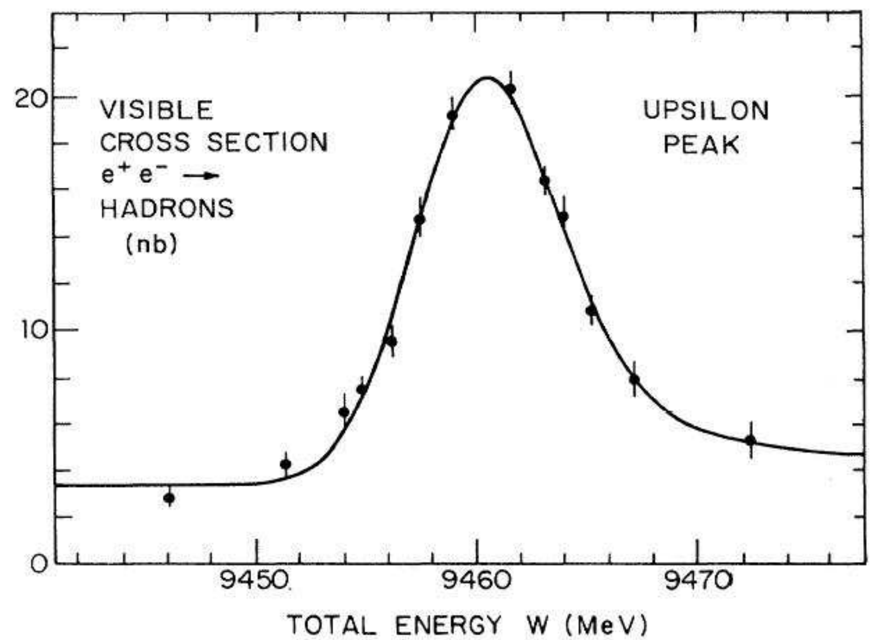
$$\frac{1}{\tau_{\rm dep}} \simeq \frac{5\sqrt{3}}{8}\,\frac{r_e\gamma^5\hbar}{m_e}\,\frac{1}{L}\oint\left\langle\frac{1}{|\rho|^3}\left(\frac{\partial\hat{n}}{\partial\delta}\right)^2\right\rangle_s ds$$

$$\frac{1}{\tau_{\rm pol}} \simeq \frac{1}{\tau_{\rm ST}} + \frac{1}{\tau_{\rm dep}}$$



Precision Mass Measurements

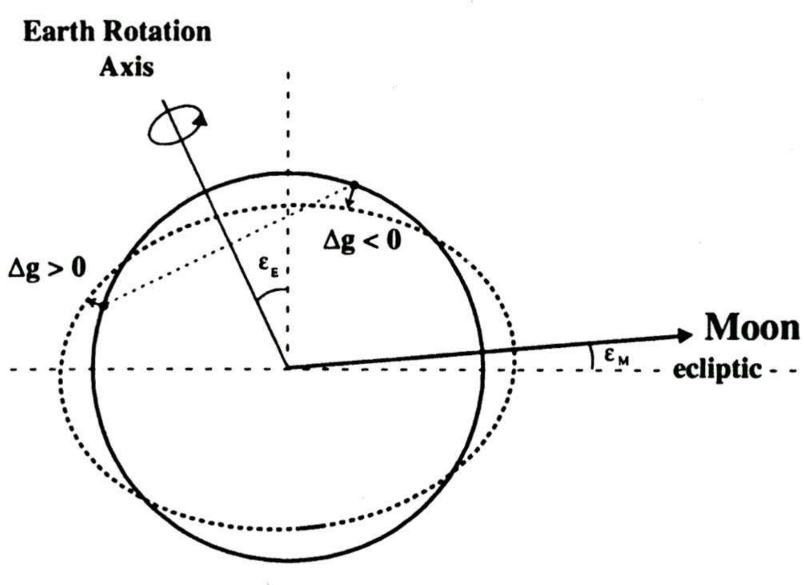


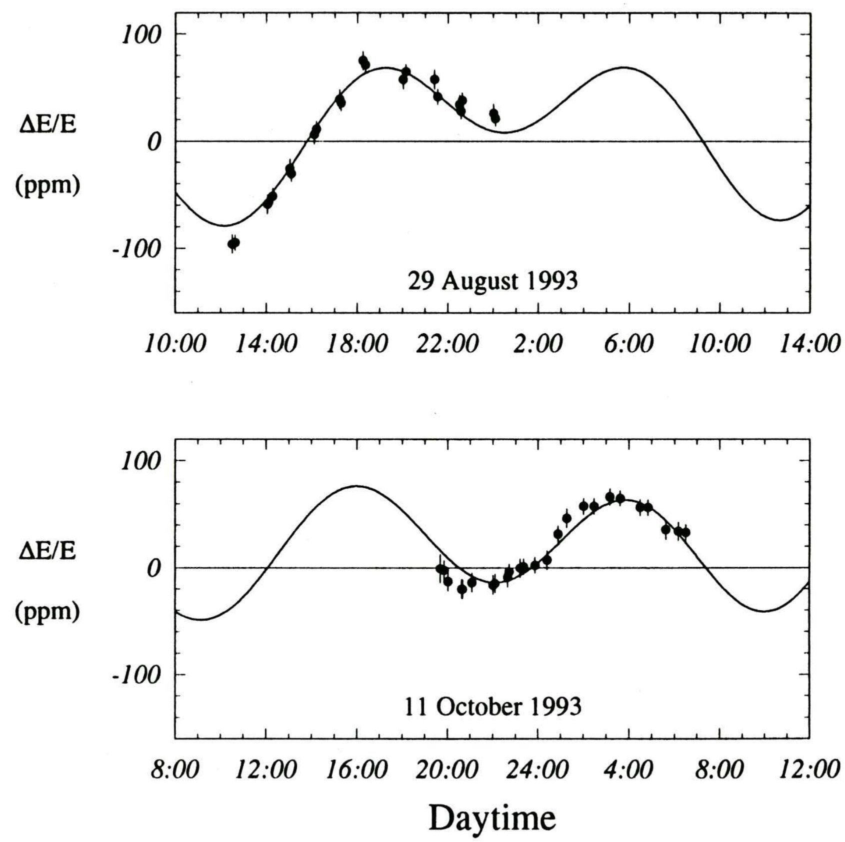


As an example from CESR (CUSB): $M_{\Upsilon} = 9459.97 \pm 0.11 \pm 0.07$ MeV

[Phys Rev D29, 2483 (1984)].

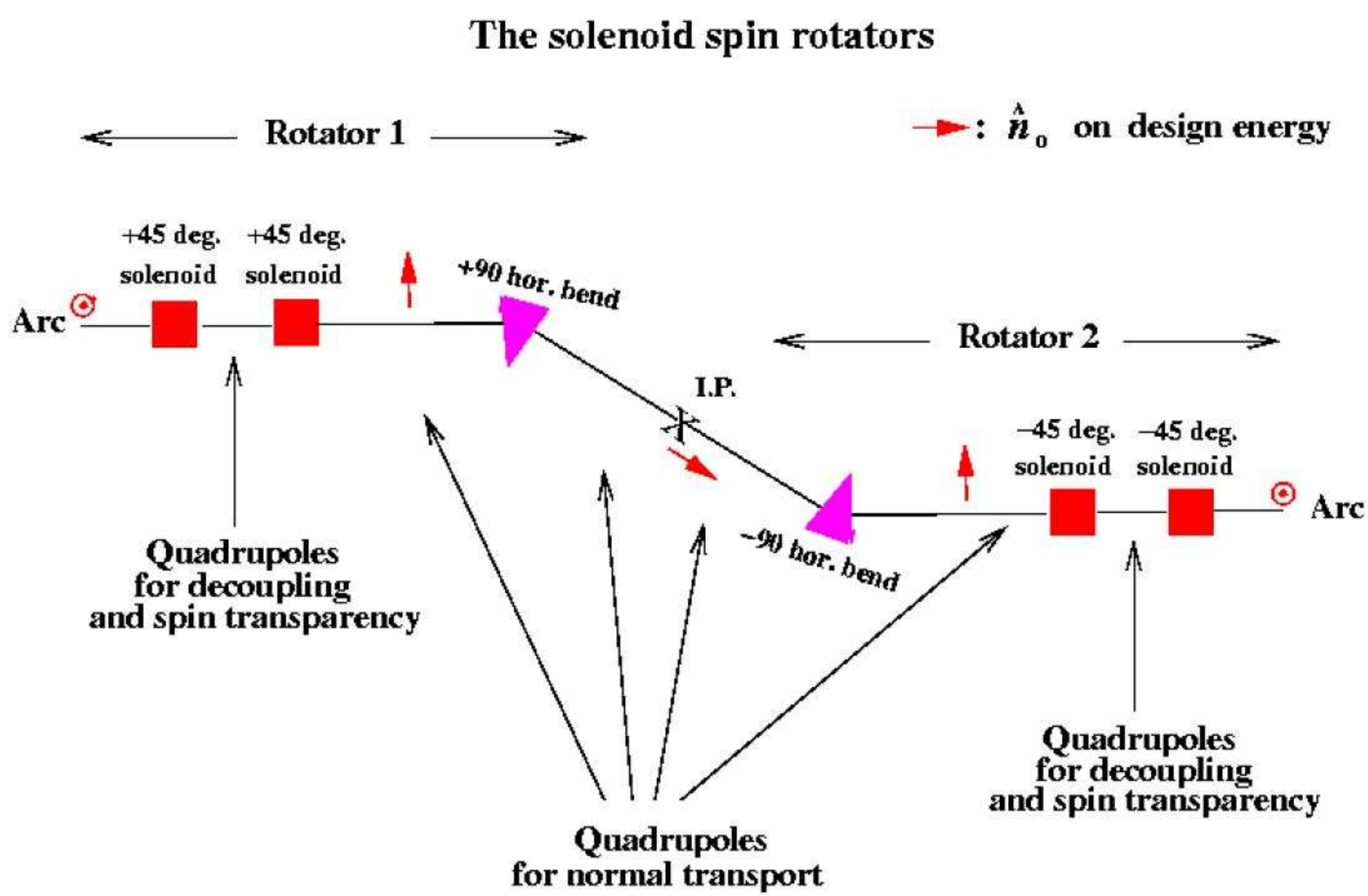
Tidal Effects at LEP





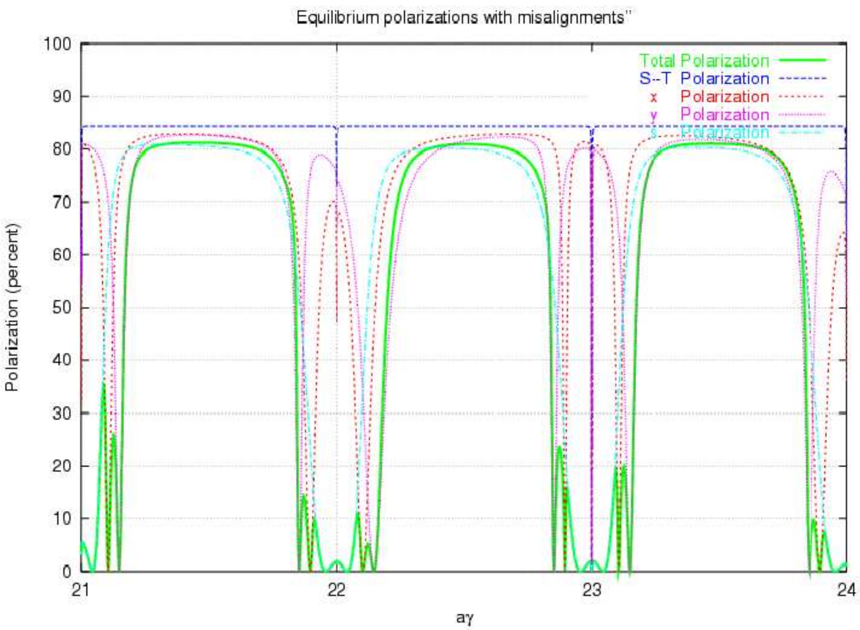
From Angelika Drees' Thesis

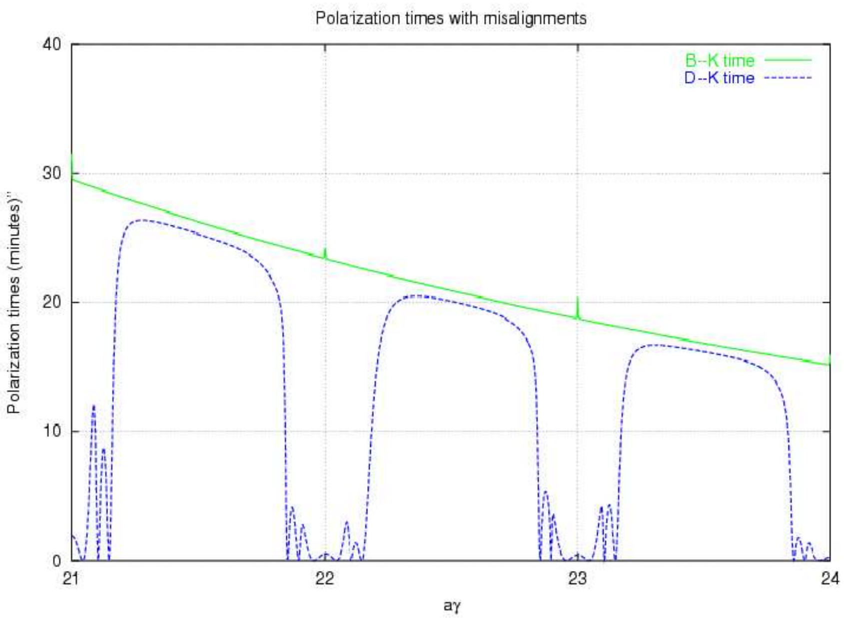
eRHIC Spin Rotators for Electrons



(Des Barber in "eRHIC Zero$^{\mathrm{th}}$-Order Design Report)

Polarization in eRHIC-e





Some References (by no means all!)

1. M. Conte and W. W. MacKay, *An Introduction to the Physics of Particle Accelerators*, World Sci., (1991).
2. L. H. Thomas, *Phil. Mag.* S. 7, **3**, 1 (1927).
3. M. H. L. Pryce, *Proc. Royal Soc. of London*, **A195**, 62 (1949).
4. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, (1972).
5. Misner, Thorne, and Wheeler, *Gravitation*, Freeman, San Francisco, (1973).
6. R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, John Wiley & Sons, (1974).
7. Ya. S. Derbenev and A. M. Kondratenko, *JETP* **37**, 968 (1973).
8. Brian W. Montague, *Phys. Rep.*, **113** (1984).
9. S. Y. Lee, *Spin Dynamics and Snakes in Synchrotrons*, World Sci., (1997).
10. K. Yokoya, DESY 86-057 (1986), and KEK Report 92-6 (1992).
11. D. P. Barber et al., DESY 98-096 (1998).

12. D. P. Barber, “Electrons are not protons”, EICA Workshop, BNL, Upton NY, BNL-52663 (Feb. 2002).
13. Y. Mori, “Present Status and Future Prospects of Optically Pumped Polarized Ion Source”, AIP Conf. Proc. 293, 151 (1994).
14. A. N. Zelenski et al., *Proc. of the 1995 Part. Accel. Conf. and Int. Conf. on High-Energy Accel.*, Dallas, 864 (1996).
15. A.A. Sokolov and I.M. Ternov, *Synchrotron Radiation*, Pergamon Press (1968).
16. V.B. Berestetskii, E.M. Landau and L.P. Lifshitz, *Quantum Electrodynamics*, Pergamon Press (1982). (Landau & Lifshitz Course on Theor. Phys., Vol. 4)
17. W. W. MacKay et al., Phys. Rev. D29, 2483 (1984).
18. J. Johnson, “SPEAR Polarization” in *Polarized Electron Acceleration and Storage* Workshop, DESY-M-82/09 (1982); S. R. Mane, Phys. Rev. A36, 120 (1987).
19. W. W. MacKay et al., Phys. Rev. **D**29, 2483 (1984).
20. K. A. Drees, Thesis, WUB-DIS 97-5, Wuppertal (1997).
21. http://www.rhichome.bnl.gov/RHIC/Spin/

22. Eds. Farkhondeh and Ptitsyn, "eRHIC Zero$^{\text{th}}$-Order Design Report", C-A/AP142 (2004).