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ACCELERATOR DEVELOPMENT DEPARTMENT

BROOKHAVEN NATIONAL LABORATORY
Associated Universities, Inc.
Upton, NY 11973

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THE LONGITUDINAL COUPLING IMPEDANCE OF A TOROIDAL VACUUM CHAMBER IN THE LOW FREQUENCY RANGE

A.G. Ruggiero

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The longitudinal coupling impedance Z_n/n at the harmonic n for a toroidal vacuum chamber with perfectly conductive walls has been calculated by several people.[1-4] For the case of a pencil beam with no transverse dimensions, that is with zero width and zero height, a useful and quite general expression is

$$\frac{Z_n}{n} = -2\pi i\beta Z_o \frac{R}{h} \times$$

$$\times \sum_{\substack{p \\ \text{odd} \geq 1}} \left[\frac{S_n(\Gamma_p R, \Gamma_p a)\ S_n(\Gamma_p b,\ \Gamma_p R)}{S_n(\Gamma_p a,\ \Gamma_p b)} + \right.$$

$$\left. + \left(\frac{\alpha_p}{\beta\Gamma_p} \right)^2 \frac{P_n(\Gamma_p R, \Gamma_p a)\ P_n(\Gamma_p b, \Gamma_p R)}{P_n(\Gamma_p a, \Gamma_p b)} \right] \tag{1}$$

where a and b are respectively the minor and major radius of the vacuum chamber which has a rectangular cross-section with height h and width w=b-a. The beam is circulating on an orbit with radius R within the vacuum chamber (a<R<b) with velocity $v=\beta c$, c being the light velocity. Z_o=377 ohm is the free space impedance. Other definitions are:

$$P_n(x,y) = I_n(x)\ K_n(y) - K_n(x)\ I_n(y) \tag{2a}$$

$$S_n(x,y) = I'_n(x)\ K'_n(y) - K'_n(x)\ I'_n(y) \tag{2b}$$

where I_n, K_n are the modified Bessel functions and

$$I'_n(x) = \frac{dI_n(x)}{dx}$$

and similarly for $K'_n(x)$. The radial propagation constant

$$\Gamma_p^2 = \left(\frac{\pi p}{h} \right)^2 - \left(\frac{\beta n}{R} \right)^2 = \alpha_p^2 - \left(\frac{\beta n}{R} \right)^2 \tag{3}$$

which is positive for $n < \pi\, pR/\beta h$. Let us denote with $n_c = \pi R/h$ the harmonic which corresponds to the lowest vacuum chamber cut off. We are interested in the estimate of the coupling impedance Z_n/n for $n < n_c$. In this case Γ_p^2 is always positive and the use of modified Bessel functions in (1) the most convenient.

Observe that the summation at the right hand side of Eq. (1) is over all odd integer values of $p \geq 1$, which corresponds to the case of the beam being located on the middle plane of the vacuum chamber.

A convenient representation of the modified Bessel functions is as follows[5]

$$I_n(x) = \frac{e^x}{\sqrt{2\pi x}} \left\{ 1 - O_1(x) + O_2(x) \right\} \tag{4a}$$

$$K_n(x) = \frac{e^{-x}}{\sqrt{2x/\pi}} \left\{ 1 + O_1(x) + O_2(x) \right\} \tag{4b}$$

$$I_n'(x) = \frac{e^x}{\sqrt{2\pi x}} \left\{ 1 - O_1'(x) + O_2'(x) \right\} \tag{4c}$$

$$K_n'(x) = \frac{e^{-x}}{\sqrt{2x/\pi}} \left\{ 1 + O_1'(x) + O_2'(x) \right\} \tag{4d}$$

where

$$O_1(x) = \sum_{k=0}^{\infty} \frac{(n, 2k+1)}{(2x)^{2k+1}} \tag{5a}$$

$$O_2(x) = \sum_{k=1}^{\infty} \frac{(n, 2k)}{(2x)^{2k}} \tag{5b}$$

$$O_1'(x) = \sum_{k=0}^{\infty} \frac{4n^2+4(2k+1)^2-1}{4n^2-(4k+1)} \frac{(n,2k+1)}{(2x)^{2k+1}} \tag{5c}$$

$$O_2'(x) = \sum_{k=1}^{\infty} \frac{4n^2+16k^2-1}{4n^2-(4k-1)^2} \frac{(n,2k)}{(2x)^{2k}} \tag{5d}$$

and

$$(n,k) = \frac{\Gamma(\frac{1}{2}+n+k)}{k!\,\Gamma(\frac{1}{2}+n-k)} \tag{6}$$

$$\Gamma(\frac{1}{2} + n) = 1.3.5.7....(2n-1)/2^n \tag{7}$$

When Eqs. (4a-d) are inserted in Eqs. (2a-b) we obtain

$$P_n(x,y) = \frac{\sinh(x-y)}{\sqrt{xy}} \; [\; 1 + T_n(x,y) \;] \tag{8a}$$

and

$$S_n(x,y) = \frac{\sinh(x-y)}{\sqrt{xy}} \; [\; 1 + T_n'(x,y) \;] \tag{8b}$$

where

$$\begin{aligned} T_n(x,y) = {} & O_2(x) + O_2(y) - O_1(x)\,O_1(y) - O_2(x)\,O_2(y) + \\ & + \Big[\, O_1(y) - O_1(x) + O_2(x)\,O_1(y) - O_1(x)\,O_2(y) \,\Big] \operatorname{cotgh}(x-y) \end{aligned} \tag{9}$$

and a similar equation for $T_n'(x,y)$ which is obtained from (9) by replacing O_1 and O_2 respectively with O_1' and O_2'.

Insertion of Eqs. (8a-b) in Eq. (1) gives

$$\frac{Z_n}{n} = 2\pi i\beta Z_o \sum_{\substack{p \\ \text{odd}\geq 1}} \frac{\sinh\Gamma_p(R-a)\sinh\Gamma_p(b-R)}{\Gamma_p h \sinh\Gamma_p(b-a)} \times$$

$$\times \left\{ \left(\frac{\alpha_p}{\beta\Gamma_p} \right)^2 \frac{\left[1 + T_n(\Gamma_p R,\Gamma_p a)\right] \cdot \left[1 + T_n(\Gamma_p b,\Gamma_p R)\right]}{1 + T_n(\Gamma_p a,\Gamma_p b)} + \right.$$

$$\left. \frac{\left[1 + T'_n(\Gamma_p R,\Gamma_p a)\right] \cdot \left[1 + T'_n(\Gamma_p b,\Gamma_p R)\right]}{1 + T'_n\ (\Gamma_p a,\Gamma_p b)} \right\} \tag{10}$$

This expression is quite general and valid for any harmonic n. The quantity between the graph parentheses is real for $n<n_c$.

If we retain only the first term in the summations at the right hand side of Eqs. (5a-d) we have for $x \to \infty$

$$O_1(x) = \frac{4n^2-1}{8x} \tag{11a}$$

$$O_2(x) = \frac{(4n^2-1)\ (4n^2-9)}{2(8x)^2} \tag{11b}$$

$$O'_1(x) = \frac{4n^2+3}{8x} \tag{11c}$$

$$O'_2(x) = \frac{(4n^2-1)\ (4n^2+15)}{2(8x)^2} \tag{11d}$$

Thus in the low frequency approximation, so that $n^2 \ll x$, all the four quantities above are small and to zero order can be neglected if compared to unit. Inspection of (9) then shows that also T_n and T'_n can be neglected within that approximation.

In this case, Eq. (10) reduces to the following form

$$\frac{Z_n}{n} = 2\pi i\beta Z_o \times$$

$$\times \sum_{\substack{p \\ \text{odd}\geq 1}} \frac{\sinh\Gamma_p(R-a)\ \sinh\Gamma_p(b-R)}{\Gamma_p h\ \sinh\ \Gamma_p(b-a)} \left\{ \left(\frac{\alpha_p}{\beta\Gamma_p} \right)^2 - 1 \right\} \tag{12}$$

In the following we shall assume that

$$\eta = \frac{w}{2R} \ll 1$$

so that

$$\Gamma_p a \sim \Gamma_p R \sim \Gamma_p b$$

and that h and w are of the same order of magnitude.

From Eq. (13)

$$\left(\frac{\alpha_p}{\beta\Gamma_p} \right)^2 - 1 = \left(\frac{\alpha_p}{\beta\gamma\Gamma_p} \right)^2 + \left(\frac{\beta n}{\Gamma_p R} \right)^2$$

where $\gamma^2 = (1-\beta^2)^{-1}$. For $n \lesssim n_c$, $\Gamma_p \sim \alpha_p$ and

$$\left(\frac{\alpha_p}{\beta\Gamma_p} \right)^2 - 1 \cong \frac{1}{\beta^2\gamma^2} + \left(\frac{\beta h n}{\pi p R} \right)^2 \tag{13}$$

which for the range of interest is a small quantity.

Observe that the condition $n^2 \ll \Gamma_p R$ under which Eq. (12) has been derived is equivalent to

$$n \lesssim \sqrt{n_c} \tag{14}$$

which is more limiting than $n \lesssim n_c$ under which Eq. (13) has been derived.

Substituting Eq. (13) in Eq. (12) gives

$$\frac{Z_n}{n} = \frac{iZ_o}{2\beta\gamma^2} G_o + iZ_o\beta \left(\frac{\beta h n}{\pi R} \right)^2 G_1 \tag{15}$$

where

$$G_o = 2 \sum_{\substack{p \\ \text{odd} \geq 1}} \frac{\text{tagh}(\alpha p)}{p} \tag{16}$$

and

$$G_1 = \sum_{\substack{p \\ \text{odd} \geq 1}} \frac{\text{tagh}(\alpha p)}{p^3} \tag{17}$$

with $\alpha = \frac{\pi w}{2h}$.

The summations (16) and (17) have been derived assuming the beam is located in the center of the vacuum chamber, that is $b-R = R-a = w/2$.

The first term at the right hand side of Eq. (15) is the usual "space charge" contribution which can be derived also in the case of a straight pipe. The second term is the contribution from the "curvature". Observe the strong dependence with the harmonic number (n^2).

For $w \gtrsim h$ $G_1 \sim 1$, whereas G_o diverges. This is due to our assumption by which the beam has no transverse dimensions. Including a beam height d with uniform distribution (but still with zero width), the summation (16) is modified as follows[4]

$$G_o = 2 \sum_{\substack{p \\ \text{odd} \geq 1}} \left(\frac{\sin up}{up} \right)^2 \frac{\text{tagh}(\alpha p)}{p} \tag{18}$$

with $u = \frac{\pi d}{2h}$.

It can be proven that for $w \gtrsim h$, G_o does not depend much on α (like G_1) and that

$$G_o = 1 + \log \frac{h}{d} .$$

Eq. (15) is valid only for harmonics satisfying (14). A closer look to Eq. (13), nevertheless, shows that

$$\left(\frac{\alpha_p}{\beta \Gamma_p} \right)^2 \sim 1$$

and that one should consider all other terms appearing in Eq. (10) that could give a first oder contribution comparable to the difference between $(\alpha_p/\beta\Gamma_p)^2$ and 1, which is the source of the "curvature" term in Eq. (15).

The following applies only for those harmonics n satisfying (14).

We shall concentrate our analysis to the quantity between graph parentheses at the right hand side of Eq. (10). We shall denote this quantity with Q. In zeroth order approximation it is given by Eq. (13).

Since T_n and T'_n are expected to be small quantities, to first order we have

$$Q = \left(\frac{\alpha_p}{\beta\Gamma_p} \right)^2 \left[1 + T_n(\Gamma_p R, \Gamma_p a) + T_n(\Gamma_p b, \Gamma_p R) + \right.$$

$$\left. - T_n(\Gamma_p a, \Gamma_p b) \right] + \tag{19}$$

$$- \left[1 + T'_n(\Gamma_p R, \Gamma_p a) + T'_n (\Gamma_p b, \Gamma_p R) - T'_n (\Gamma_p a, \Gamma_p b) \right] .$$

It is sufficient to retain only the lowest order contributions, that is Eqs. (11a-d). Thus, taking into account that $x \sim y$ and letting $y = x(1+\delta)$ with $|\delta| \ll 1$, we have from Eq. (9) to lowest order and $n \gg 1$:

$$T_n(x,y) = - \frac{n^2}{2x^2} \left[1 + (y-x) \right] + \ldots \tag{20a}$$

$$T'_n(x,y) = \frac{n^2}{2x^2} \left[1 - (y-x) \right] + \ldots \tag{20b}$$

Insertion of these equations in (19) gives

$$Q = \left(\frac{\alpha_p}{\beta\gamma\Gamma_p} \right)^2 + \left(\frac{\beta n}{\Gamma_p R} \right)^2 +$$

$$- \left(\frac{n}{\Gamma_p R} \right)^2 + \frac{n^2 w/R}{\Gamma_p R} \left(\frac{\beta n}{\Gamma_p R} \right)^2 + \ldots \tag{21}$$

The last terms are the contributions we were looking for to the first order approximation. Thus

$$Q = \frac{1}{\beta^2\gamma^2} - \left(\frac{hn}{\pi_p R} \right)^2 \left(\frac{1}{\gamma^2} - \frac{\beta^2 n^2 wh}{\pi p R^2} \right) \tag{22}$$

and

$$\frac{Z_n}{n} = \frac{iZ_o}{2\beta\gamma^2} G_o - i \frac{Z_o\beta}{\gamma^2} \left(\frac{hn}{\pi R}\right)^2 G_1 +$$

$$+ iZ_o\beta \left(\frac{hn}{\pi R}\right)^2 \left(\frac{\beta n\sqrt{wh/\pi}}{R}\right)^2 G_2 \tag{23}$$

where

$$G_2 = \sum_{\substack{p \\ \text{odd} \geq 1}} \frac{\text{tgh}(\alpha p)}{p^4} \tag{24}$$

Eq. (23) is more accurate than (15). The last two terms are the contributions from the "curvature". The first has the γ^{-2} dependence like the "space-charge" term but with opposite sign. For this term there is still a "cancellation" between electric and magnetic fields. The second term is nevertheless that part of the magnetic field that cannot be "cancelled" properly.

We have derived Eq. (23) for $n \lesssim \sqrt{n_c}$, and we cannot prove at the moment if its validity holds beyond this range. Otherwise for $n \sim n_c$ the last term of Eq. (23) would give a predominant contribution. Observe that $G_2 \sim G_1 \sim 1$.

References

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