Chapter 7

Radio Frequency Basics and Superconductivity

7.1 Introduction

An radio frequency (RF) cavity is a resonant waveguide with closed boundaries. Microwave power in the RF cavity is coupled to the particle beam to accelerate the particles to high energies. It is of interest to find monochromatic waves propagating down the waveguide of the form

$$\vec{E} = \vec{E}_0 e^{i(kz-\omega t)} \tag{7.1}$$

$$\vec{B} = \vec{B}_0 e^{i(kz-\omega t)} \tag{7.2}$$

The $\vec{E}$ and $\vec{B}$ fields must satisfy Maxwell's equations inside the waveguide given by

$$\begin{array}{ll} \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, & \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}, & \vec{\nabla} \cdot \vec{E} = 0 \end{array} \tag{7.3}$$

Combining Maxwell's equations, we arrive at two uncoupled wave equations

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \left\{ \begin{array}{c} \vec{E} \\ \vec{B} \end{array} \right\} = 0 \tag{7.4}$$

The $\vec{E}$ and $\vec{B}$ fields can be determined by solving the eigenvalue equation subject to boundary conditions

$$\hat{n} \times \vec{E} = 0, \quad \hat{n} \cdot \vec{B} = 0 \tag{7.5}$$

The solutions to the eigenvalue problem by substituting Eq. 7.1 and 7.2 form an orthogonal set of eigenvalues each with unique frequency and corresponding field configuration. Since, the electric and magnetic fields are independent with different boundary conditions, they form two families of solutions which are classified as transverse magnetic (TM) and transverse electric (TE) modes.

7.2 Pill-Box Cavity

In a pill-box cavity the fields are additionally subject to Eq. 7.5 at $z = 0$ and $z = l$, where l is the length of the cylindrical cavity. Reflections at the z-boundaries create appropriate standing waves of the form

$$\vec{E} = \vec{E}_0 \cos\left(\frac{p\pi z}{l}\right) e^{i\omega t}, \quad p = 0, 1, 2 \ldots \text{(TM Modes)} \tag{7.6}$$

$$\vec{B} = \vec{B}_0 \sin\left(\frac{p\pi z}{l}\right) e^{i\omega t}, \quad p = 1, 2, 3 \ldots \text{(TE Modes)} \tag{7.7}$$

Substituting the above fields into the eigenvalue equation, two families of modes similar to a waveguide are obtained. These modes are classified as TM_{mnp} or TE_{mnp}, where m, n, and p are integers and describe the azimuthal, radial, and longitudinal periodicity. The resonant frequencies of TM or TE modes are given by

$$\omega_{mnp} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \sqrt{\left(\frac{p_{mn}}{r}\right)^2 + \left(\frac{p\pi}{l}\right)^2} \; - \; \text{(TM)} \tag{7.8}$$

$$\omega_{mnp} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \sqrt{\left(\frac{p'_{mn}}{r}\right)^2 + \left(\frac{p\pi}{l}\right)^2} \; - \; \text{(TE)} \tag{7.9}$$

where, p_{mn} and p'_{mn} are the nth zero of the Bessel function and its derivative respectively, and r is the radius of the cylinder. The mode frequencies as a function of cavity dimensions for a pill box cavity are shown in Fig. 7.1

To accelerate particles, a longitudinal $\vec{E}$ is required which is satisfied only by J_0. The lowest accelerating mode of the type TM_{0np} has fields of the form

$$E_z = E_0 J_0(\omega_0 r/c) \cos(\omega_0 t) \tag{7.10}$$

$$H_\phi = -\frac{1}{\mu_0 c} E_0 J_0(\omega_0 r/c) \sin(\omega_0 t) \tag{7.11}$$

with all other field components are zero. This mode is denoted as TM_{010} with fields similar to that shown in Fig. 7.2. The frequency of this mode in a pill-box cavity is independent of the cavity length and is given by

$$\omega = \frac{2.405\ c}{R} \tag{7.12}$$

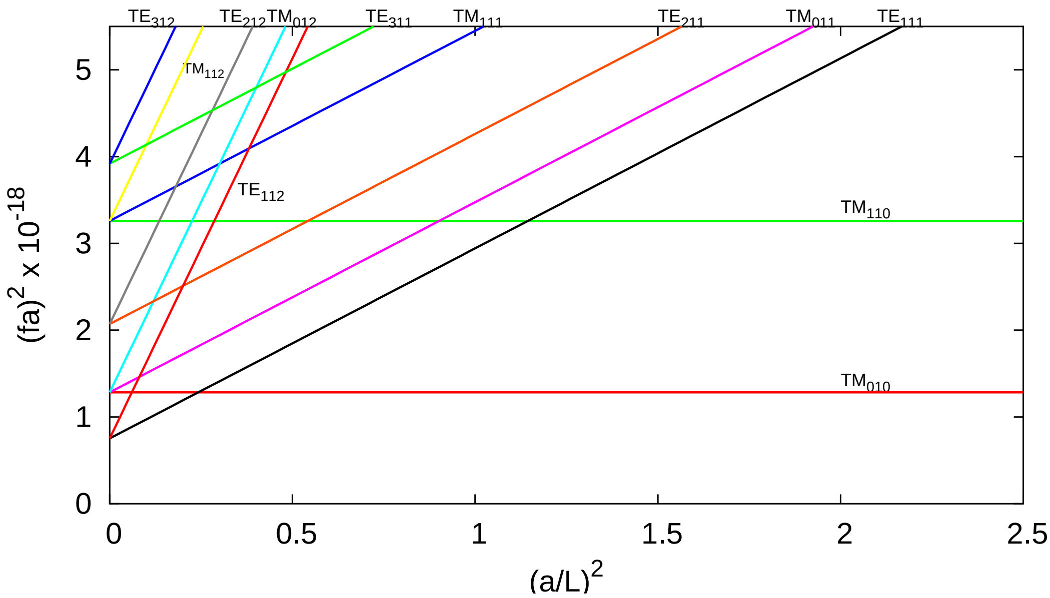

Figure 7.1: Mode frequencies as a function of cavity dimension for a pill box resonator.

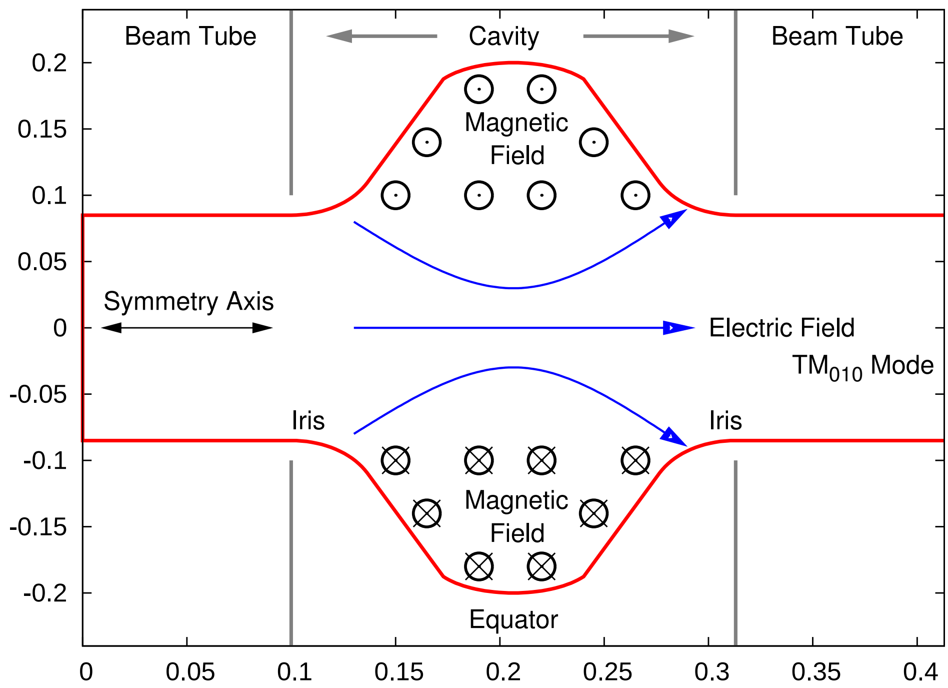

Figure 7.2: Schematic of an elliptical cavity with field lines of TM_{010} mode.

7.3 Characteristic Parameters

Some relevant figures of merit characterizing RF cavities are described in the following section.

7.3.1 Accelerating Voltage

The longitudinal electric field accelerates or decelerates a particle depending on the particle phase with respect to the RF. The voltage gained by the particle across the gap is given by

$$V_{acc} = \left| \int\limits_{z=0}^{z=l} E_z e^{i\omega_0 z/c} dz \right| \tag{7.13}$$

where l is the cavity length, and ω_0 is the frequency of the mode. The particle takes a finite time to traverse the cavity, leading to a reduction in energy gain which is characterized by a transit time factor

$$T = \frac{\int_0^l E_0 e^{i\omega z/c}\, dz}{\int_0^l E_0\, dz} \tag{7.14}$$

7.3.2 Stored Energy

For each mode in the cavity, the time averaged energy in the electric is equal to that in the magnetic fields. The stored energy in the fields is given by a volume integral

$$U = \frac{1}{2}\epsilon_0 \int\limits_V |\vec{E}|^2\, dv = \frac{1}{2}\mu_0 \int\limits_V |\vec{H}|^2\, dv \tag{7.15}$$

7.3.3 Surface Resistance and Power Dissipation

Since, real metals have finite conductivity, RF fields at sufficiently high frequencies penetrates a finite depth into the conductor. The current density is exponentially decreasing into the metal. The surface impedance of the metal is given by

$$R_s = \sqrt{\frac{\mu\omega}{2\sigma}} \tag{7.16}$$

where, δ is known as the skin depth which is defined as

$$\delta = \sqrt{\frac{2}{\mu\omega\sigma}}. \tag{7.17}$$

The power dissipated in the cavity walls due to the surface resistance is given by the surface integral

$$P_d = \frac{1}{2} R_s \int_S |\vec{H}|^2 \, ds \tag{7.18}$$

7.3.4 Quality Factor

The stored energy in the cavity decays exponentially

$$U(t) = U_0 \; e^{-t/\tau} \tag{7.19}$$

where τ is the characteristic time constant dependent on the material. This figure of merit can also be expressed in terms of a quality factor

$$Q_0 = \frac{\omega_0 U(t)}{P_d(t)} \tag{7.20}$$

which characterizes the amount of stored energy dissipated in on RF cycle

7.3.5 Geometric Factor and Shunt Impedance

A product of the surface resistance and quality factor is geometric constant given by

$$G = R_s Q_0 = \frac{\omega_0 \mu_0 \int_V |\vec{H}^2 dv}{\int_S |\vec{H}|^2 ds} \tag{7.21}$$

Another figure of merit of a cavity which is the shunt impedance given by

$$R_{shunt} = \frac{V^2}{P_d} \tag{7.22}$$

which measures the efficiency the accelerating voltage for given dissipation. A more meaningful quantity is the ratio of shunt impedance to the quality factor

$$\frac{R_a}{Q_0} = \frac{V^2}{\omega U} \tag{7.23}$$

This quantity is independent of the cavity material and the field amplitude and is a measure of the efficiency of the accelerating voltage for a given stored energy.

7.4 RF Superconductivity

The two most important characteristics of superconducting RF (SRF) cavities are the high average gradient and cavity intrinsic Q_0. The fundamental advantage of superconducting materials is due to the extremely small surface resistance (< 10 nΩ) compared to conventional copper which is typically several orders higher ($\sim$ mΩ). Therefore, operation of cavities in CW mode or high duty factor become feasible due to significant reduction of power dissipation on the cavity walls. A simple example comparing the power consumption of SRF cavities to copper cavities is illustrated in Table 7.1 [63]

Table 7.1: AC power required to operate 500 MHz superconducting and normal conducting cavities at 1 MV/m

Option	SRF	Normal
Frequency [MHz]	500	500
Q_0	2×10^9	2×10^4
R_a/Q_0 [Ω/m]	330	900
P/L [W/m]	1.5	5.6×10^4
AC Power [kW/m]	0.54	112

Limitations on the maximum power dissipated on the cavity walls resulting in vacuum degradation, stresses, and metal fatigue puts an upper limit on the maximum gradient. In addition, SRF cavities also offer the option of having a larger beam pipe to reduce wakefield effects. However, this results in a drop in the R/Q_0, but the very large intrinsic Q_0 factor naturally helps to compensate for this drop compared to copper cavities.

7.4.1 Superconductivity

The unique properties of superconductors has been subject of scrutiny for several decades. A simplified two-fluid model proposed by London offers a phenomenological explanation of the field exclusion of below the critical temperature. The zero dc resistivity of a superconductor can be attributed to the fact that super-electrons carry the current while shielding the field from the normal electrons.

A more successfully microscopic theory to explain superconductivity was put forth by Bardeen, Cooper, and Schrieer (BCS) in 1957. At a temperature below the critical temperature (T_c), it is energetically favorable for a fraction of the electrons to be paired into Cooper pairs. The number of unpaired electrons is

$$n_{normal} \propto e^{-\frac{\Delta}{k_B T}} \tag{7.24}$$

where, k_B is the Boltzmann factor. These pairs consist of opposite spin and momenta electrons which freely move without resistance. At $T = 0$ K, all charge carriers are condensed into a single state transforming the metal into an ideal superconductor.

7.4.2 Surface Resistance of Superconductor

Although, the dc resistivity is zero, there are small losses in the presence of RF currents. The Cooper pairs posses inertia and at microwave frequencies they are unable to screen the external fields completely from the normal electrons. Although, Eq. 7.24 predicts that the surface resistance goes to zero at $T = 0$ K, measurements reveal

$$R_s = A\frac{1}{T}f^2 e^{-\frac{\Delta(T)}{k_B T}} + R_0 \tag{7.25}$$

where Δ is the half the energy needed to break a Cooper pair, and A is a material dependent constant which is dependent on the penetration depth, coherence length, Fermi velocity, and the mean free path. R_0 is known as the residual resistance and can be affected by trapped flux and surface impurities. The operating temperature of the SRF cavities is usually chosen in the range where the BCS resistance dominates and the Carnot efficiency is tolerable.

7.4.3 Critical Fields

When considering superconducting materials for practical applications (for example SRF cavities), the maximum tolerable surface fields are of substantial interest. The ultimate theoretical limit is only posed by RF critical magnetic field beyond which the screening effect of the Cooper pairs is lost. Existing superconductors are classified as either Type I or II and differ in the surface energy between the normal and superconducting boundaries.

Type I superconductors have positive surface energy and remain in a Meisnner state up to a temperature dependent critical field H_c which is given by [64]

$$H_c(T) = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right] \tag{7.26}$$

Type II superconductors such as niobium are different because of their negative surface energy of the interface. Therefore, it is energetically favorable for the flux to create an interface. Beyond a critical field H_{cl}, a mixed state of superconductor and normal conducting zones in periodic lattice is created which also referred to as vortices. The flux in the vortices is given by

$$\Phi_0 = \frac{hc}{2e} \tag{7.27}$$

The densities of the vortices increases with external magnetic field until the entire sample is normal conducting at a second critical field H_{c2}. For niobium, the choice of metal for SRF cavities, the theoretical critical field limit is approximately 2200-2400 Oe.

There are no theoretical limitations on the RF electric fields, but accelerating cavities are usually operated well below field level ($\sim$ 150 MV/m) to support the critical magnetic field. This is due to several reasons like thermal breakdown originating from surface defects, resonant electron multiplication or multipacting, field emission, and other phenomena which are studied in great detail in Refs. [63, 65].

7.4.4 Elliptical Multi-cell Cavities

The phenomenon of resonant electron multiplication in RF electric fields or multipacting results in absorption of RF power and eventually breakdown superconducting cavities. A more detailed treatment of multipacting can be found in Ref. [63]. The most successful solution proposed to reduce or eliminate multipacting was to use a spherical geometry to force the charges to drift to the equator [66]. At the equator, $E_\perp$ vanishes thereby reducing the number of secondaries and suppress multipacting. Modern SRF cavities are elliptical in shape due to mechanical stability requirements and ease of chemical treatment of the cavity surface compared to the spherical shape [67].

The quest for achieving higher gradients to maximize the "real-estate" gradient (or "MV/m") has lead to strongly coupled multi-cell cavities. These cavities can be viewed as coupled oscillators where each mode in a single cell cavity is split into a passband of normal modes. The length of each cell is chosen to be $\lambda/2$ so that the acceleration in the multi-cell cavity is maximized for the eigenmode in the TM_{010} passband with π phase advance between each cell.