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COUPLING AND DECOUPLING IN STORAGE RINGS

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INTRODUCTION

The accidental or intentional introduction of solenoids and skew quadrupoles into a storage ring couples the transverse motion, unless the lattice is trimmed. Coupling can cause, for example, irreversible emittance blow up at injection (protons), resonance excitation, flat beam blow up in collision (electrons), and can destructively modify the beam-beam equations of motion. Apart from these effects themselves, not many diagnostics are available to guide compensation attempts.

After developing a matrix description of coupling, this paper analyses the compensation of a lattice insertion, and goes on to show how random errors can be handled close to a coupling resonance. The global decoupling of random errors in the SPS, achieved by the observation of eigenfrequencies on a spectrum analyser, is described. Rough estimates show that global compensation usually makes further (local) compensation unnecessary.

MATRIX FORMALISM

It is always possible to linearise the equations of transverse motion of a particle about its equilibrium orbit, even if this is distorted and passes through non-linear elements like sextupoles. Motion from i to j is described by

$$1)\quad Y_j = S_{ji} Y_i$$

where S is a 4x4 transfer matrix, and Y represents the physical displacement of the particle from the equilibrium orbit. It is more convenient to work in a normalized coordinate system X, such that

$$2)\quad X_i = G_i Y_i = \begin{pmatrix} G_x & o \\ o & G_z \end{pmatrix}_i Y_i$$

where, for example, the G_x matrix is

$$3)\quad G_x = \begin{pmatrix} \frac{1}{\sqrt{\beta_x}} & 0 \\ \frac{\alpha_x}{\sqrt{\beta_x}} & \sqrt{\beta_x} \end{pmatrix}$$

Transfer from i to j is now described by T_{ji},

$$4)\quad X_j = T_{ji} X_i = (G_j S_{ji} G_i^{-1}) . G_i Y_i$$

In the ideal uncoupled machine T_{ji} simply rotates particles in X-phase space through the difference in betatron phases, so it is written in terms of the rotation matrix R as

$$5)\quad T_{ji} = \begin{pmatrix} R(\phi_{xj} - \phi_{xi}) & 0 \\ 0 & R(\phi_{zj} - \phi_{zi}) \end{pmatrix}$$

The ideal one turn matrix at P is written as

$$6)\quad T_p = \begin{pmatrix} R_x & 0 \\ 0 & R_z \end{pmatrix}$$

while in general Tp is coupled, and written as

$$7)\quad T_p = \begin{pmatrix} M & m \\ n & N \end{pmatrix}$$

In a lattice coupled by thin skew quadrupoles of focal length f, and thick solenoids of angular twist θ, it has been shown [1] that Tp is approximated to first order in coupler strength by

$$8)\quad T_p \approx \begin{pmatrix} R_x & 0 \\ 0 & R_z \end{pmatrix} \begin{pmatrix} I & F \\ -F^+ & I \end{pmatrix} = \begin{pmatrix} R_x & R_x F \\ -R_z F^+ & R_z \end{pmatrix}$$

Here the first order fundamental 'projection' matrix F is given (with $\phi p=0$) by

$$9)\quad F = \sum_{\substack{\text{skew} \\ \text{quads}}} q\, R(-\phi_x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} R(\phi_z) + \sum_{\text{soln}} \theta\, R(-\phi_x) G_x G_z^{-1} R(\phi_z)$$

where the dimensionless skew quad strength is

$$10)\quad q = \frac{\sqrt{\beta_x \beta_z}}{f}$$

The summation in 9) is over all downstream couplers, at positive phase displacements from the reference point. A superscript plus sign $^+$ is used to denote the adjoint operation,

$$11)\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xleftrightarrow{+} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

so that the adjoint of a unimodular 2x2 matrix is its own inverse. A comparison of 7) and 8) shows that the matrices m and n are first order in coupler strength, but that M and N are only perturbed in higher orders.

Edwards and Teng [2,3] have shown that matrices U and V always exist such that

$$12)\quad T = VUV^{-1} = \begin{pmatrix} I\cos\psi & D\sin\psi \\ -D^+\sin\psi & I\cos\psi \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I\cos\psi & -D\sin\psi \\ D^+\sin\psi & I\cos\psi \end{pmatrix}$$

where A, B and D are unimodular. The normal modes of the motion correspond to the eigenvectors of A and B, so that the displacement on turn n is just

$$13)\quad X_n = VU^n . (V^{-1} X_o)$$

If the matrix V is written in component form as

$$14)\quad V = \begin{pmatrix} c & o & sd & se \\ o & c & sf & sg \\ -sg & se & c & o \\ sf & -sd & o & c \end{pmatrix} \qquad \begin{array}{l} c = \cos\psi \\ s = \sin\psi \\ dg - fe = 1 \end{array}$$

then the transverse positions at the reference point P, as a function of time, become

$$15)\quad \begin{array}{l} x = a_1 c.c_1 + a_2 s.(dc_2 + es_2) \\ z = a_1 s.(-gc_1 + es_1) + a_2 c.c_2 \\ c_1 = \cos(2\pi Q_1 n + \phi_1) \text{ etc.} \end{array}$$

Here the amplitudes (emittances) a_1, a_2, and the phases ϕ_1, ϕ_2, are constants of the motion, while Q_1 and Q_2 are the eigenfrequencies of A and B. This motion is easy to visualise for the special cases shown in Figure 1.

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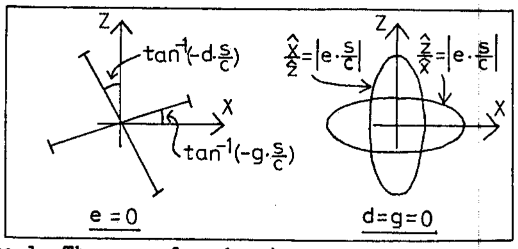

Figure 1. The normal modes in two special cases

Three useful time averages, which follow directly from 15),

$$16)\quad \begin{aligned} 2\langle x^2\rangle &= a_1{}^2c^2 + a_2{}^2s^2(d^2+e^2) \\ 2\langle z^2\rangle &= a_1{}^2s^2(g^2+e^2) + a_2{}^2c^2 \\ 2\langle xz\rangle &= -a_1{}^2\, scg + a_2{}^2\, sc\, d \end{aligned}$$

show, for example, that flat beams, ($a_1 >> a_2$), are vulnerable to vertical blow up through the 'e' component. If V refers to the collision point of flat beams, luminosity is lost unless

$$17)\quad e\ \sin\psi << \sqrt{\frac{\epsilon_V}{\epsilon_H}}$$

If D is a unit matrix, the angle ψ measures the angle of normal mode twist, and there is no distortion of the beam shape. D is given by

$$18)\quad D = \frac{H}{\det^{\frac{1}{2}}(H)}$$

where the fundamental coupling matrix H is

$$19)\quad H = m + n^+ = R_xF - FR_z^+$$

The angle ψ is a generally convenient measure of the contortion of the normal modes, and is given in terms of 'known' quantities by

$$20)\quad \tan(2\psi) = \frac{-\det^{\frac{1}{2}}(H)}{\frac{1}{2}\mathrm{Tr}(M-N)}$$

Close to coupling resonances, when Trace(M-N) passes through zero, ψ reaches extreme values of +/- $\pi/4$.

The eigenfrequency splitting is given by

$$21)\quad (\cos(2\pi Q_1)-\cos(2\pi Q_2))^2 = (\tfrac{1}{2}\,\mathrm{Tr}(M-N))^2 + \det(H)$$

and is always more than the design tune split, Q_x-Q_z. The closest approach of Q_1 and Q_2 is one measure of the coupling in a lattice. These last two equations lead to the geometrical interpretation of Figure 2.

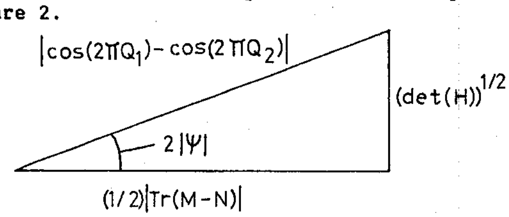

Figure 2. Geometric relation of tune splits and ψ

A COUPLING INSERTION

Solenoidal magnetic detectors are often included in storage rings, usually compensated locally by skew quadrupoles. The strengths of these skew quads must be chosen so that the matrix F (and H) dissapears everywhere outside the insertion. Evaluating F at a point with phases $-\Delta\phi_x$ and $-\Delta\phi_z$,

$$22)\quad F_\Delta = R(-\Delta\phi_x)F_*R(\Delta\phi_z)$$

and the insertion is decoupled if the matrix F_* is identically zero, that is, with $\phi_* = 0$, if

$$23)\quad F_* = \sum q\, R(-\phi_x)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}R(\phi_z) + \theta\begin{pmatrix}\sqrt{\frac{\beta_z^*}{\beta_x^*}} & 0\\ 0 & \sqrt{\frac{\beta_x^*}{\beta_z^*}}\end{pmatrix} = 0$$

The solenoid has been assumed to lie symmetrically about simultaneous beta minima.

Four simultaneous equations have to be satisfied, requiring at least four skew quads. Supposing that the lattice and the coupler layout is symmetric about $\phi_*=0$, F_{*21} and F_{*12} are identically zero if the symmetric skew quad pairs are antisymmetrically powered, that is if

$$24)\quad q(-\phi) = -q(\phi)$$

Although it is convenient to write these equations in terms of betatron functions and phases, global parameters, insertion decoupling is completely independent of outside magnet geometries and currents. Conspicuously absent from 23) are the lattice tunes Q_x and Q_z.

A complete solution of the problem must show that the perturbation of the normal modes inside the insertion is not harmful. It can be rigourously shown, and is intuitively obvious, that V_* at the crossing point in a decoupled machine is

$$25)\quad V_* \approx \begin{pmatrix} I & F_{*u}\\ -F_{*u}^+ & I\end{pmatrix}$$

where the matrix F_{*u} includes only upstream couplers in the insertion. Continuing the example of a symmetric insertion, F_{*u} is in general

$$26)\quad F_{*u} = \sum_{\phi<0} q\begin{pmatrix} 0 & -\sin\phi_x\,\sin\phi_z\\ \cos\phi_x\,\cos\phi_z & 0\end{pmatrix}$$

This shows that there is no normal mode twisting, but that flat beams are blown up vertically, unless the upper right matrix element is reduced to zero. It has also been shown[1], theoretically and experimentally, that this element destructively modifies the beam-beam behaviour of flat beams. The remaining non-zero coupling matrix element is innocuous. For these reasons, insertions in electron machines should include three pairs of skew quadrupoles.

RANDOM COUPLING ERRORS

The matrix H

It will now be assumed that only accidental coupling remains in the lattice, due for example to twisted quadrupoles, or to vertical orbit errors in sextupoles. Also, the fractional tunes are assumed to have a fairly small design tune split, ΔQ, as in the SPS and many other rings,

$$27)\quad Q_{x,z} = Q_o \pm \tfrac{1}{2}\Delta Q$$

justifying the approximations for R_x and R_z

$$28)\quad R_{x,z} \approx R_o \pm \pi\Delta Q.R_o'$$

Keeping two terms in a Taylor expansion of H,

29) $$H \approx H_o + \pi\Delta Q . H_1$$

then, using 19),

30) $$H_o = \sum_{SQ} q\, R(-\phi_x)\left[R_o\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}R_o^+\right]R(\phi_z) + \sum_{SOLN} \theta\, R(-\phi_x)\left[R_o G_x G_z^{-1} - G_x G_z^{-1} R_o^+\right]R(\phi_z)$$

and

31) $$H_1 = \sum_{SQ} qR(-\phi_x)\left[R_o'\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}R_o'\right]R(\phi_z) + \sum_{SOLN} \theta\, R(-\phi_x)\left[R_o' G_x G_z^{-1} + G_x G_z^{-1} R_o'\right]R(\phi_z)$$

where some rotation matrices have been commuted.

Global decoupling

The matrix H_0 is readily simplified to

32) $$H_o = \sin(2\pi Q_o)\left[\sum_{SQ} q\, R(\phi_z - \phi_x) + \sum_{SOLN} g\theta R(\phi_z - \phi_x + \omega)\right]$$

where the solenoid weight factor g is

33) $$g = (\gamma_x\beta_z + \gamma_z\beta_x + 2(1 - \alpha_x\alpha_z))^{\frac{1}{2}}$$

and the solenoid angle ω is

34) $$\tan(\omega) = (\beta_x + \beta_z)/(\alpha_x\beta_z - \alpha_z\beta_x)$$

For an experimental solenoid g is 2, and ω is $\pi/2$. H_0 may also be described by a vector, since 32) could also be written

35) $$H_o = \sin(2\pi Q_o)\begin{pmatrix}p & r\\ -r & p\end{pmatrix}$$

where

36) $$\begin{pmatrix}p\\ r\end{pmatrix} = \sum_{SQ} q\begin{pmatrix}\cos(\phi_z - \phi_x)\\ \sin(\phi_z - \phi_x)\end{pmatrix} + \sum_{SOLN} g\,\theta\begin{pmatrix}\cos(\phi_z - \phi_x + \omega)\\ \sin(\phi_z - \phi_x + \omega)\end{pmatrix}$$

When the reference point P (the phase origin) at which H_0 has been calculated is moved azimuthally, this vector rotates, but explicitly preserves its length. The smallest normal mode splitting, δQ, is measured by this length, since from 21),

37) $$\delta Q \approx \frac{1}{2\pi}\,\frac{\det^{\frac{1}{2}}(H_o)}{\sin(2\pi Q_o)} = \frac{1}{2\pi}\,(p^2 + r^2)^{\frac{1}{2}}$$

Global decoupling, making $H_0=0$, is achieved in principle by adjusting two skew quad strengths until δQ is reduced to zero.

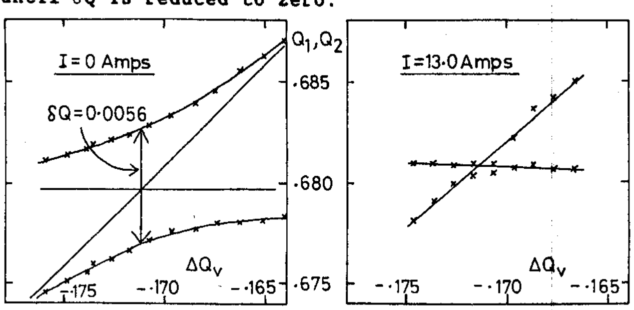

Figure 3. SPS eigenfrequencies, skew quad on/off

Figure 3 shows the eigenfrequencies in the SPS as a function of the design vertical tune, with and without a current of 13.0 Amps in one skew quad. This quad has a fortunate phase location, since it alone removes almost all of the natural SPS coupling.

Decoupling at finite ΔQ

The matrix H_1 can also be simplified

38) $$H_1 = H_o \cot(2\pi Q_o) - F.2\sin(2\pi Q_o)$$

but, due to the F contribution, it is not simply behaved azimuthally. The different behavior of H_0 and H_1 is illustrated by the example of a lattice with a decoupled insertion. Since the transfer matrix across the insertion is unchanged to first order in coupler strength, then the design tunes are the real tunes, and H_0 is zero everywhere, even inside the insertion. Outside the insertion, the normal modes are not perturbed for any tunes, so F and H_1 also dissapear. Inside the insertion, however, the normal modes can be very contorted, meaning that F and H_1 may be very large.

After making $H_0=0$, the coupling angle is

39) $$\tan(2|\psi|) = \frac{\det^{\frac{1}{2}}(H)}{2\pi\Delta Q\,\sin(2\pi Q_o)} \approx \det^{\frac{1}{2}}(F)$$

Both ψ and D are now independent of the separation in tunes, an important result. If desired, further decoupling is possible at places such as injection, extraction, or collision points, by local control of F. Rough estimates of H_0 and H_1 in a statistical model suggest that this is often not necessary.

For example, take a lattice with 4Q regular cells, with a sextupole at each quadrupole, with random quad twists of $\langle\theta^2\rangle^{\frac{1}{2}}$, and with vertical orbit errors of $\langle z^2\rangle^{\frac{1}{2}}$. The components of H_0 are expected to be

40) $$\langle p^2\rangle^{\frac{1}{2}} = \langle r^2\rangle^{\frac{1}{2}} \approx \frac{1}{2}(8Q)^{\frac{1}{2}}\left[2\langle\theta^2\rangle^{\frac{1}{2}} + \frac{\langle z^2\rangle^{\frac{1}{2}}}{\langle\eta\rangle}\right] q_o$$

where $\langle\eta\rangle$ is a typical dispersion, say R/Q^2, and q_0 is the strength of a regular quadrupole, about 2 in dimensionless units. A rough estimate of the minimum tune split is then

41) $$\langle(\delta Q)^2\rangle^{\frac{1}{2}} \approx \frac{1}{2\pi}\,(8Q)^{\frac{1}{2}}\left[2\langle\theta^2\rangle^{\frac{1}{2}} + \langle z^2\rangle^{\frac{1}{2}}\,\frac{Q^2}{R}\right]$$

The natural SPS coupling due to sextupoles, with $\langle z^2\rangle^{\frac{1}{2}} = 3$ mm, Q = 27, R = 1100 m, and $\langle\theta^2\rangle^{\frac{1}{2}} = 0$, is then estimated to be about $\delta Q = 4.6\times10^{-3}$, surprisingly close to the observed value of 5.6×10^{-3}.

Comparing 9) and 36), the expected size of the matrix elements of F, and H_1, can be written in terms of the expected minimum tune split. Putting this into 39), local decoupling is not necessary after global decoupling if angles of magnitude

42) $$|\psi| \approx \pi\,\delta Q$$

are acceptable.

Acknowledgements

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References

1) S. Peggs, CERN/SPS/82-2
2) D. Edwards and L. Teng, IEEE Trans.Nucl.Sci,NS-20,No.3,1973.
3) L. Teng, NAL Report FN-229, 1971.