

# NEW FORMALISM IN THE SPIN TRACKING CODE SPINK \*

A.U. Luccio, F. Lin,  
Brookhaven National Laboratory, Upton, NY

## Abstract

The code Spink[1], in use for many years to track polarized hadrons in a synchrotron, was overhauled with the introduction of a new generalized Frenet-Serret system of coordinates in all dimensions in space.

## THE BMT EQUATION

For spin tracking in a synchrotron *Spink* expresses the spin rotation in each machine element in matrix form

$$\mathbf{S} = \mathcal{M} \mathbf{S}_0. \quad (1)$$

Start from the BMT equation (no electric field)

$$\begin{cases} \frac{d\mathbf{S}}{dt} = \frac{q}{\gamma m} \mathbf{S} \times \mathbf{F} \\ \mathbf{F} = (1 + G\gamma)\mathbf{B}_\perp + (1 + G)\mathbf{B}_\parallel \end{cases}, \quad (2)$$

using the magnetic field components transverse and longitudinal with respect to the velocity  $\mathbf{v}$ . Define a unit velocity  $\mathbf{u} = \mathbf{v}/v$ ,  $v = \beta c$ . and use vector identities to express the magnetic field *components* in terms of the magnetic field  $\mathbf{B}$

$$\mathbf{B}_\perp = (\mathbf{u} \times \mathbf{B}) \times \mathbf{u}, \quad \mathbf{B}_\parallel = (\mathbf{u} \cdot \mathbf{B})\mathbf{u}. \quad (3)$$

After some readjustment ( $\mathbf{B} = \mathbf{B}_\parallel + \mathbf{B}_\perp$ )

$$\mathbf{F} = (1 + G\gamma)\mathbf{B} - G(\gamma - 1)(\mathbf{u} \cdot \mathbf{B})\mathbf{u}. \quad (4)$$

Use a coordinate system ( $\mathbf{e}$ ) that revolves around the accelerator. The axis  $\hat{z}$ , longitudinal, is tangent to the trajectory,  $\hat{x}$  and  $\hat{y}$  are the displacements with respect to this orbit, radial and vertical, respectively. Call  $s$  the longitudinal coordinate along the orbit. In these coordinates the derivative of a vector  $\mathbf{a}$  is

$$\frac{d\mathbf{a}}{ds} = \frac{da_x}{ds}\hat{x} + a_x \frac{d\hat{x}}{ds} + \frac{da_y}{ds}\hat{y} + a_y \frac{d\hat{y}}{ds} + \frac{da_z}{ds}\hat{z} + a_z \frac{d\hat{z}}{ds}. \quad (5)$$

Since it is

$$\frac{q}{\gamma m} = \frac{v_e}{B\rho},$$

where  $B$  is the local field,  $\rho$  the local radius of curvature,  $v_e$  the velocity in the new coordinates. Write

$$\frac{1}{\rho_x} = \frac{\sin \theta}{\rho}, \quad \frac{1}{\rho_y} = \frac{\cos \theta \sin \phi}{\rho},$$

and obtain

$$\begin{aligned} \frac{d\hat{x}}{ds} &= \frac{\sin \theta}{\rho} \hat{z}, & \frac{d\hat{y}}{ds} &= -\frac{\cos \theta \sin \phi}{\rho} \hat{z} \\ \frac{d\hat{z}}{ds} &= -\frac{\sin \theta}{\rho} \hat{x} + \frac{\cos \theta \sin \phi}{\rho} \hat{y} \end{aligned}. \quad (6)$$

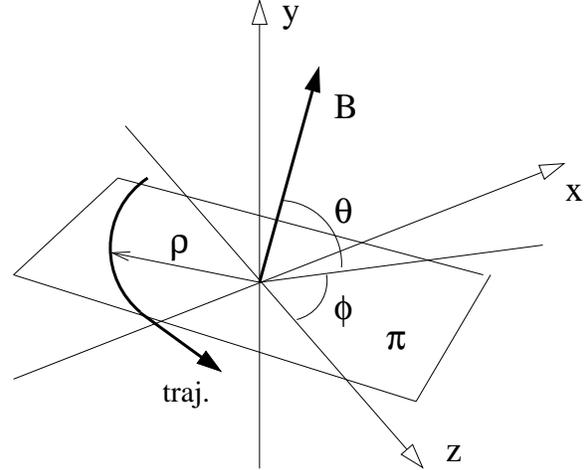


Figure 1: Particle trajectory contained in a plane ( $\pi$ ) perpendicular to the axis  $\mathbf{B}$ .  $\theta$  is the angle of latitude between  $\mathbf{B}$  and the “horizontal” plane ( $x, z$ ). ( $\phi$ ) is the angle of longitude with respect to the axis  $\hat{z}$ .

$\sin \theta/\rho$  and  $\cos \theta \sin \phi/\rho$  are the local curvature of the particle orbit in two planes, respectively, see Fig.1.

The velocity in the new coordinates at  $z = 0$  is obtained using Eqs.(5) and (6), see also S.Y.Lee book[2]

$$\frac{d\mathbf{r}}{dt} = \left[ x' \hat{x} + y' \hat{y} + \left( z' + \frac{x}{\rho} \sin \theta - \frac{y}{\rho} \cos \theta \sin \phi \right) \hat{z} \right] \frac{ds}{dt}. \quad (7)$$

Primes denote derivative with respect to  $s$ . The longitudinal component of the velocity is

$$z' = \sqrt{1 - x'^2 - y'^2}.$$

We will use the coordinate transformation Eq.(5) and Eq.(7) to express Eq.(2) for spin precession in a thin accelerator element in matrix form as in Eq.(1). Assume that within the element the magnetic field and the orbit don't change, perhaps slicing each element and writing matrices for each slice.

Rewrite Eq.(2) in the new variable  $s$  as

$$\frac{d\mathbf{S}}{ds} = \mathbf{S} \times \mathbf{f} \quad (8)$$

where

$$\mathbf{f} = \frac{h}{B\rho} [(1 + G\gamma)\mathbf{B} - G(\gamma - 1)(\mathbf{u} \cdot \mathbf{B})\mathbf{u}], \quad (9)$$

with

$$h = \frac{v_e}{v} = \sqrt{x'^2 + y'^2 + \left( z' + \frac{x}{\rho} \sin \theta - \frac{y}{\rho} \cos \theta \sin \phi \right)^2}$$

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Use Eq.(5) for the derivative of a vector

$$\frac{d\mathbf{S}}{ds} = \left( S'_x - S_z \frac{\sin\theta}{\rho} \right) \hat{x} + \left( S'_y + S_z \frac{\cos\theta \sin\phi}{\rho} \right) \hat{y} + \left( S'_z + S_x \frac{\sin\theta}{\rho} - S_y \frac{\cos\theta \sin\phi}{\rho} \right) \hat{z},$$

that, in view of Eq.(8), is equivalent to three scalar differential equations for the spin vector components

$$\begin{cases} S'_x = f_z S_y - \left( f_y - \frac{\sin\theta}{\rho} \right) S_z \\ S'_y = -f_z S_x + \left( f_x - \frac{\cos\theta \sin\phi}{\rho} \right) S_z \\ S'_z = \left( f_y - \frac{\sin\theta}{\rho} \right) S_x - \left( f_x - \frac{\cos\theta \sin\phi}{\rho} \right) S_y \end{cases} \quad (10)$$

In the assumed thin machine element the system (10) yields three 3.rd order formally identical linear equations for the three components of spin

$$\begin{cases} S'''' + \omega^2 S' = 0, \text{ with} \\ \omega^2 = \left( f_x - \frac{\cos\theta \sin\phi}{\rho} \right)^2 + \left( f_y - \frac{\sin\theta}{\rho} \right)^2 + f_z^2. \end{cases} \quad (11)$$

The general integral of Eq.(11) is

$$S = C_1 + C_2 \cos \mu + C_3 \sin \mu$$

with the angle of spin precession around the local axis

$$\mu = \omega \delta s.$$

$\delta s$  is the trajectory path length through the element.

The result matrix  $\mathcal{M}$  for Eq.(1) is

$$\begin{pmatrix} 1 - (\mathcal{B}^2 + \mathcal{C}^2)c & \mathcal{A}\mathcal{B}c + \mathcal{C}s & \mathcal{A}\mathcal{C}c - \mathcal{B}s \\ \mathcal{A}\mathcal{B}c - \mathcal{C}s & 1 - (\mathcal{A}^2 + \mathcal{C}^2)c & \mathcal{B}\mathcal{C}c + \mathcal{A}s \\ \mathcal{A}\mathcal{C}c + \mathcal{B}s & \mathcal{B}\mathcal{C}c - \mathcal{A}s & 1 - (\mathcal{A}^2 + \mathcal{B}^2)c \end{pmatrix}, \quad \cos\theta = B_x/B, \quad \sin\theta = B_y/B, \quad B = \sqrt{B_x^2 + B_y^2}, \quad (12)$$

with:  $c = 1 - \cos \mu$ ,  $s = \sin \mu$  and

$$\begin{cases} \mathcal{A} = \frac{1}{\omega} \left( f_x - \frac{\cos\theta \sin\phi}{\rho} \right), \\ \mathcal{B} = \frac{1}{\omega} \left( f_y - \frac{\sin\theta}{\rho} \right), \quad \mathcal{C} = \frac{1}{\omega} f_z \end{cases}.$$

The determinant of  $\mathcal{M}$  is

$$\det(\mathcal{M}) = 1 - 2\mathcal{D}c + \mathcal{D}^2 c^2 + \mathcal{D}s^2, \quad \mathcal{D} = \mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2.$$

As it should, it is  $\det(\mathcal{M}) = 1$ , since  $\mathcal{D} = 1$ .

## PARTICULAR CASES

Expand  $h$  to first order in the position and velocity using  $\mathbf{f}$  of Eq.(9)

$$h \approx 1 + \frac{x}{\rho} \sin\theta - \frac{y}{\rho} \cos\theta \sin\phi.$$

## Horizontal bend

The field (axis of rotation) is vertical, along  $\hat{y}$ . It is

$$B_x = 0, \quad B_y \equiv B \neq 0, \quad B_z = 0, \quad \theta = \pi/2, \quad \phi = 0,$$

$$\frac{h}{B\rho} = \frac{q}{m\gamma v}$$

$$f_x = 0, \quad f_y - \frac{1}{\rho} = \frac{G\gamma}{\rho} - \frac{1 + G\gamma}{\rho} \frac{x}{\rho}, \quad f_z = 0.$$

The spin rotation angle around the vertical is

$$\mu = \omega \delta s = \frac{q}{m\gamma} \left[ G\gamma \left( 1 + \frac{x}{\rho} \right) + \frac{x}{\rho} \right] B \delta s,$$

To lowest order the spin rotation angle is proportional to the integrated magnetic field with a coefficient  $G\gamma$ .

## Quadrupole

The strength of a quadrupole is defined through the parameter  $K_1$  and the beam rigidity  $pc/q$

$$K_1 = -\frac{\partial B/\partial r}{pc/c}.$$

The magnetic field, at the particle transverse location in the laboratory coordinates, has components

$$B_x = K_1(pc/q)y, \quad B_y = K_1(pc/q)x, \quad B_z = 0.$$

The axis angles are  $\theta$ , and  $\phi$  (assume)  $= \pi/2$

$$\cos\theta = B_x/B, \quad \sin\theta = B_y/B, \quad B = \sqrt{B_x^2 + B_y^2},$$

the curvature parameter is  $h/(B\rho) = q/(m\gamma v)$ , and

$$\begin{cases} f_x - \frac{\cos\theta}{\rho} = \frac{\cos\theta}{\rho} [G\gamma + (1 + G\gamma) o(1)] \\ f_y - \frac{\sin\theta}{\rho} = \frac{\sin\theta}{\rho} [G\gamma + (1 + G\gamma) o(1)] \end{cases},$$

with

$$o(1) = \frac{x \sin\theta + y \cos\theta}{\rho}$$

After quadrature and root extraction, only retaining first order terms in the transverse position of the beam, the angle kick is then

$$\mu = \omega \delta s = \frac{q}{m\gamma} \{ G\gamma [1 + o(1)] + (1 + G\gamma) o(1) \} B \delta s.$$

To lowest order the spin rotation angle in a quadrupole, and for the matter in any element with transverse field, is proportional to the integrated field with a coefficient  $G\gamma$

### Radio frequency dipole. Horizontal

Only the radial component of the magnetic field is present

$$B_x = B \neq 0, \quad B_y = B_z = 0.$$

In this device the radial field  $\mathbf{B}$  is oscillatory, producing a vertical oscillation of the beam that modulates the vertical betatron oscillation. When the frequency of  $\mathbf{B}$  is varied, a spin resonance condition can be reached that generates a spin flip.

The instantaneous vertical orbit kick is

$$\delta p_y = \frac{B \delta s}{pc/c} \cos \Phi, \quad \Phi = 2\pi \int f dt,$$

with  $\Phi$  a phase angle and  $f$  the modulation frequency,

The preceding formalism for the horizontal bend applies, just exchanging the rôles between  $f_x$  and  $f_y$ . The resulting spin precession angle kick is, to leading order

$$\mu = G\gamma \delta p_y.$$

### Solenoid - no fringe

Only the longitudinal component of the magnetic field does not vanish

$$B_x = B_y = 0, \quad B_z \equiv B \neq 0,$$

also the trajectory is not bent:  $1/\rho = 0$ . To lowest order the spin rotation angle is

$$\mu = \omega \delta s = \frac{q}{m\gamma} (1 + G) B \delta s,$$

### General rotation matrix

For a rotation axis set to any direction in space, with an angle  $\theta$ , latitude, with respect the horizontal plane  $\hat{x}$ ,  $\hat{z}$  and an angle  $\phi$ , longitude, with respect to the longitudinal axis, for the general rotation matrix of Eq.(12) it is

$$\mathcal{A} = \cos \theta \sin \phi, \quad \mathcal{B} = \sin \theta, \quad \mathcal{C} = \cos \theta \cos \phi$$

As before

$$c = 1 - \cos \mu, \quad s = \sin \mu,$$

with  $\mu$  the angle of spin rotation.

We use this formalism to characterize elements like thin lens Siberian snakes. E.g. in RHIC full snakes, we have  $\theta = 0$ ,  $\phi = 45^\circ$ , and  $\mu = 180^\circ$ , to indicate that the axis of the snake is in the horizontal plane, makes a 45 degrees angle with the longitudinal axis, and rotates the spin by  $180^\circ$ . In the AGS partial snakes the axis is longitudinal  $\phi = 180^\circ$ , and the rotation of the spin is about  $30^\circ$ .

## CONCLUSIONS AND ACKNOWLEDGMENTS

This remake of an old note [3] was done when we realized that at the time only the horizontal curvature of the

orbit was considered in the algorithms for *Spink*, which did not appreciably affect simulation of high energy polarized protons and therefore did not attract much attention.

This was also pointed out by Sateesh Mane who stressed that in the general theory by Kondratenko [4] as well as in his work [5] and [6] for spin motion there was always reference to the actual orbit of the particle in space.

Ernest Courant [7] also showed how one can arrive at the right formulation by integrating together the BMT equation for spin precession and the Lorentz trajectory equation.

W.Waldo MacKay [8] stressed in particular how for a correct formalism for spin precession it is important to consider the BMT equation in its original covariant form.

Andreas Lehrach motivated much of this work by the need to correctly interpret deuteron polarization measurements at COSY [9]. See a companion paper by ours in this Conference.

We had extended discussions on algorithms for spin tracking with Vahid Ranjbar of TechX Corp., who is a partner in systematic work aimed at making *Spink* an even more useful and fiendly tool for the spin community of accelerator physicists.

## REFERENCES

- [1] A.U.LUCCIO: *Spin Tracking in RHIC- Code SPINK*. In: Y.ONEL, N.PAVER, A.PENZO, editor: *Proc. Adriatico Research Conf on Trends in Collider Spin Physics. Trieste, Italy, 12/5-8, 1995*. World Scientific, p.235.
- [2] S.Y.LEE: *Spin Dynamics and Snakes in Synchrotrons*. World Scientific, Singapore, 1997.
- [3] A.U.LUCCIO: *Spin Rotation Matrices for Spin Tracking*. Technical Report BNL-62371, Brookhaven National Laboratory. Upton, NY, Oct.30 1995.
- [4] A.M.KONDRATENKO, M.A.KONDRATENKO, YU.N.FILATOV: *Analysis of the Experimental Data with Polarized Deuteron Beam in COSY*. Technical Report GOO "Zariat, Novosibirsk and JNR, Dubna, 20 Nov 2007, Translated by V.S.Morozov and M.A.Leonova.
- [5] S.R.MANE, YU, M SHATUNOV and YOKOYA K: *Spin-polarized charged particle beams in high-energy accelerators*. Reports on Progress in Physics, (68):1997–2265, 2005.
- [6] S.R.MANE: *Deuteron spin-flip resonance widths and the spin response function*. Phys.Rev.Special Topics- Accelerators and Beams, (10,111001), 2007.
- [7] E.D.COURANT: *Revised Spin Motion Equations*. Technical Report C-A/AP/292, Brookhaven National Laboratory. Upton, NY, Upton, NY.
- [8] W.W.MACKAY: *Prospects for Acceleration of Deuterons and Helions*. Technical Report C-A/AP/296, Brookhaven National Laboratory. Upton, NY, November 2007.
- [9] A.D.KRISCH, M.A.LEONOVA, V.S.MOROZOV R.S.RAYMOND.D.W.SIVERS V.K.WONG R.GEBEL A LEHRACH B.LORENTZ R.MAIER D.PRASUHN H.STOCKHORST F.HINTERBERGER and K.ULBRICH: *Unexpected reduction of rf spin resonance strength for stored deuteron beams*. Phys.Rev.Spec.Topics-Accel.and Beams, (10):07100–1, 2007.