N-turn Hamiltonians for Slow Extraction

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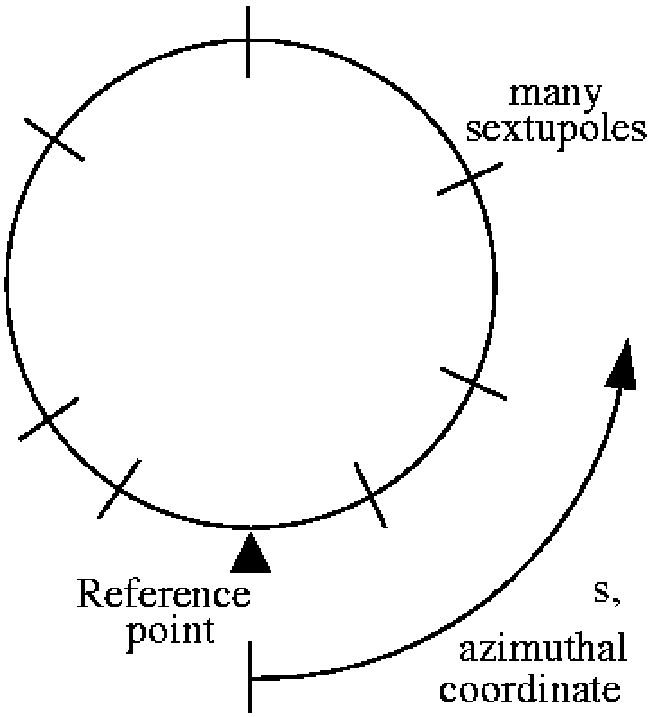
Introduction

- **Traditional / formal approach:**

$$H_C(x, x', s) \;=\; 2\pi Q_0 \,\frac{1}{2}(x^2 + x'^2) \;+\; \sum_{s_i} g_i x^3 \delta(s - s_i) \quad (1)$$

$$\begin{pmatrix} \dfrac{dx}{ds} \\ \\ \dfrac{dx'}{ds} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial H_C}{dx'} \\ \\ -\dfrac{\partial H_C}{dx} \end{pmatrix} \quad (2)$$



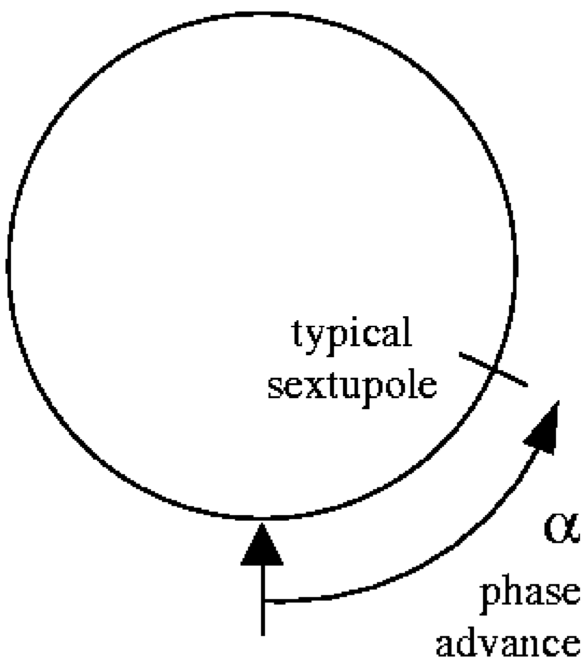
- **Time, s, is continuous**
- **H_C is not conserved**
- **How to handle the delta function?**
- **What use is this formalism?**

Difference Hamiltonians

- **Consider the Projected motion due to one sextupole:**

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\rm sext} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\rm ref} = R \begin{pmatrix} x \\ x' \end{pmatrix}_{\rm ref} \tag{3}$$

$$Projection = R^{-1}\, kick\, R \tag{4}$$



- **The net motion is first order in g .....**

$$\begin{pmatrix} \Delta x \\ \Delta x' \end{pmatrix} = \begin{pmatrix} -s\, g(c\, x + s\, x')^2 \\ c\, g(c\, x + s\, x')^2 \end{pmatrix} \tag{5}$$

- **This is succinctly described by a Projection Hamiltonian, H_P ....**

$$H_P \quad = \quad -\frac{g}{3}(c\,x + s\,x')^3 \qquad (6)$$

- **... that is shorthand for the DIFFERENCE motion**

$$\begin{pmatrix} \Delta x \\ \\ \Delta x' \end{pmatrix} = \begin{pmatrix} \frac{\partial H_P}{dx} \\ \\ -\frac{\partial H_P}{dx'} \end{pmatrix} \qquad (7)$$

- **Time is NO LONGER continuous!**
- **If desired, H_P can be cast in action-angle coordinates**

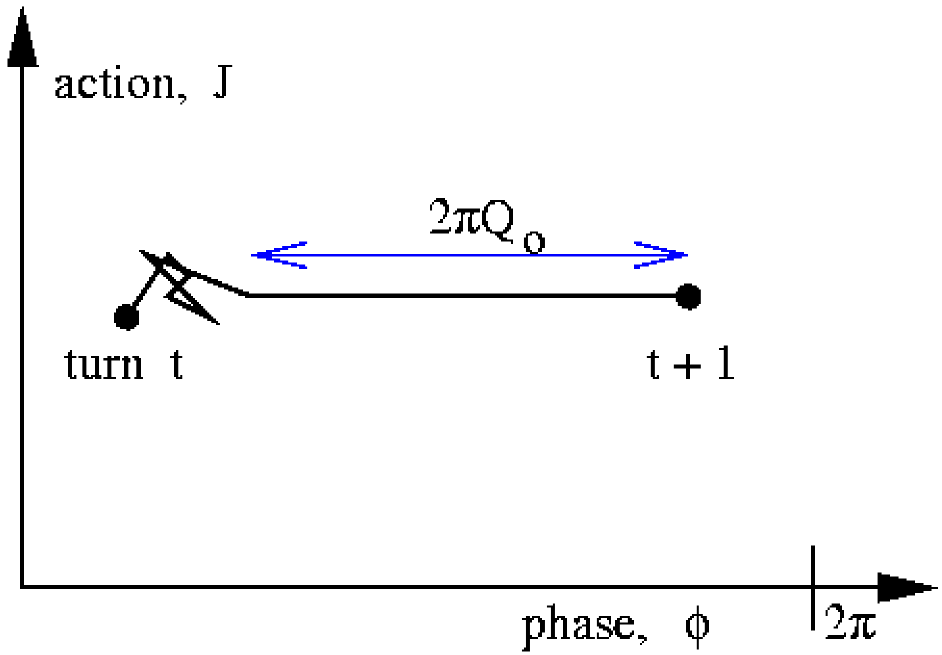
$$H_P \quad = \quad \frac{g}{3\sqrt{2}}\, J^{3/2}\, \left[\sin(3(\alpha+\phi)) - 3\, \sin(\alpha+\phi)\right] \qquad (8)$$

- **Extend to one-turn motion with many sextupoles ....**

$$H_1 \;=\; 2\pi Q_0\, J \;+\; \sum_{\mathrm{sexts}} H_P(J,\phi) \qquad (9)$$

- **... where the One-turn Difference Hamiltonian H_1 is defined by**

$$\begin{pmatrix} J \\ \\ \phi \end{pmatrix}_{t+1} \;\equiv\; \begin{pmatrix} J \\ \\ \phi \end{pmatrix}_{t} \;+\; \begin{pmatrix} -\dfrac{\partial H_1}{d\phi} \\ \\ \dfrac{\partial H_1}{dJ} \end{pmatrix}_{t} \qquad (10)$$



- **The net motion is "large"**
- **H_1 is not conserved**
- **What use is this formalism?**

- **Aside:** **this formalism easily extends to 2-D (and other multipoles) -**

  it just gets messier !

$$
\begin{aligned}
H_1 \;&=\; 2\pi Q_{x0}\, J_x \;+\; 2\pi Q_{z0}\, J_z \qquad (11) \\
&+ \sum_{\text{i,j,k,l}} V_{ijkl}\, J_x^{i/2}\, J_z^{j/2}\, \sin(k\phi_x + l\phi_z + \phi_{ijkl})
\end{aligned}
$$

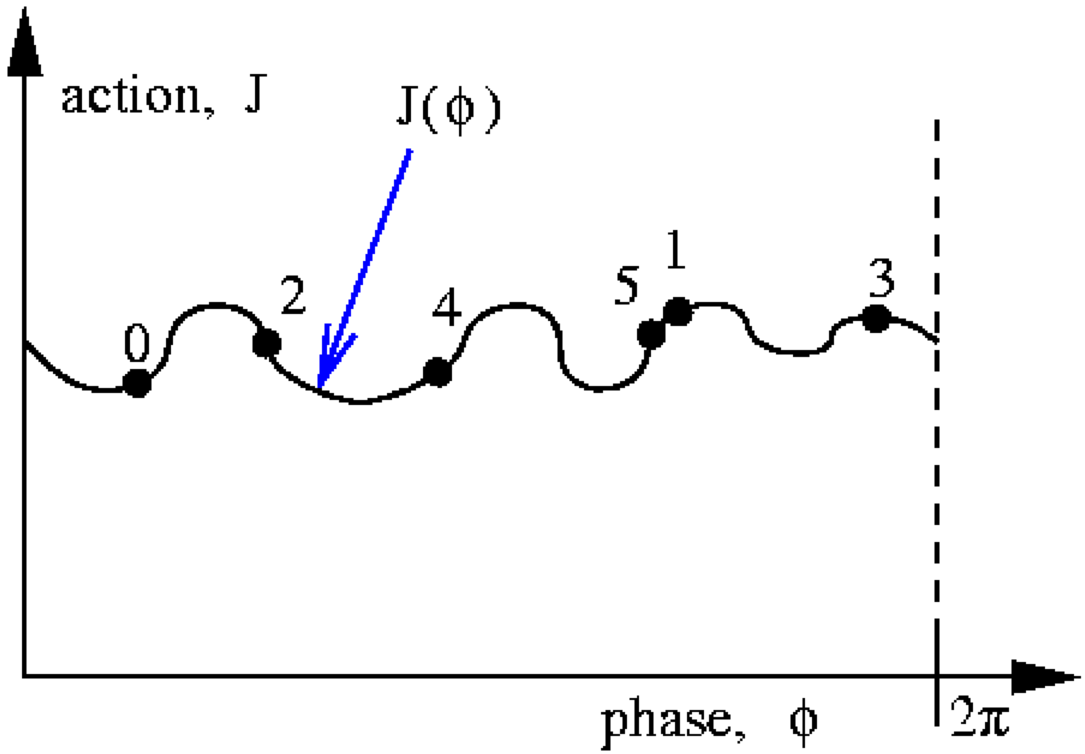
- **The above expressions works in general -**

  except in the presence of resonance islands, and/or chaos !

Use 1: "Distortion functions"

- **The KAM surface of section is solved in general by**

$$J(\phi) \;=\; J_0 \;-\; \sum_{i,k} \left(\frac{k\,V_{ik}}{2\sin(\pi k Q_0)}\right)\; J_0^{i/2}\;\sin(k\phi+\phi_{ik}) \quad (12)$$



- **Determine free variables V_{ik} and ϕ_{ik} by**
  - **- design**
  - **- simulation/tracking**
  - **- beam measurement**

Use 2: Observable time series

- **Similarly, the equation of motion is solved by**

$$J(t) \;=\; J_0 - \sum_{i,k} \left(\frac{k\, V_{ik}}{2\sin(\pi k Q_0)} \right) J_0^{i/2} \; \sin(2\pi k Q t + \phi_{0ik}) \tag{13}$$

- **This time series is directly observable, in simulation or real life ...**

$$\begin{pmatrix} x_{BPM1} \\ \\ x_{BPM2} \end{pmatrix}_{t\,=\,0,1,\,\cdots\,10^6} \longrightarrow \begin{pmatrix} J \\ \\ \phi \end{pmatrix}_{t\,=\,0,1,\,\cdots\,10^6} \tag{14}$$

- **... although it helps if you have an AC Dipole !**

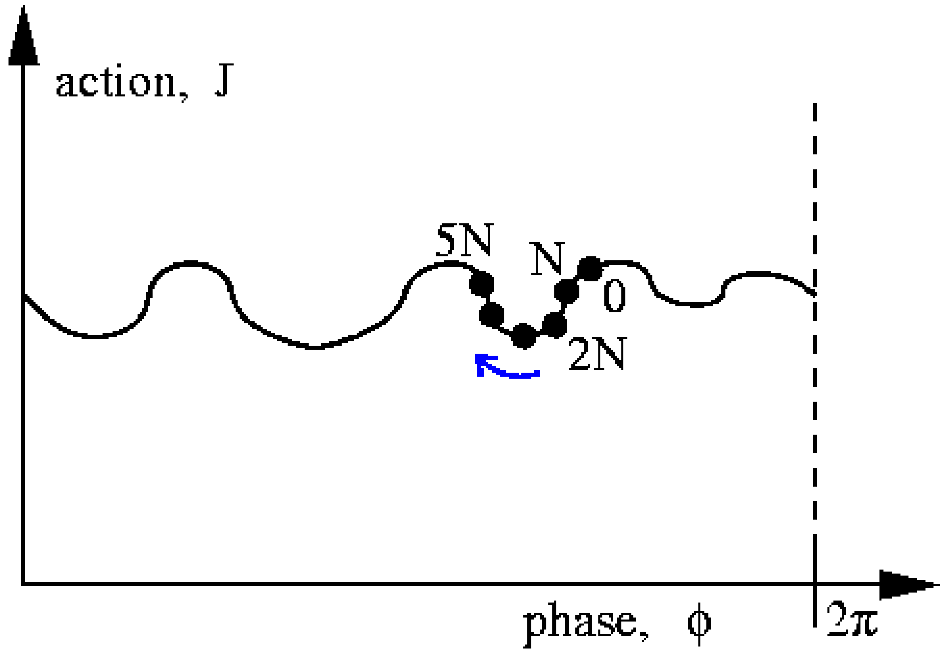
N-turn Hamiltonians

- **It is useful to extend the formalism from "1-turn" to "N-turn" ....**

$$H_N = 2\pi(Q_0 - \frac{K}{N})J + \sum_{i,k} V_{Nik} J^{i/2} \sin(k\phi + \phi_{Nik}) \quad (15)$$

$$\begin{pmatrix} J \\ \\ \phi \end{pmatrix}_{t+N} \equiv \begin{pmatrix} J \\ \\ \phi \end{pmatrix}_t + \frac{1}{N} \begin{pmatrix} -\frac{\partial H_1}{d\phi} \\ \\ \frac{\partial H_1}{dJ} \end{pmatrix}_t \quad (16)$$

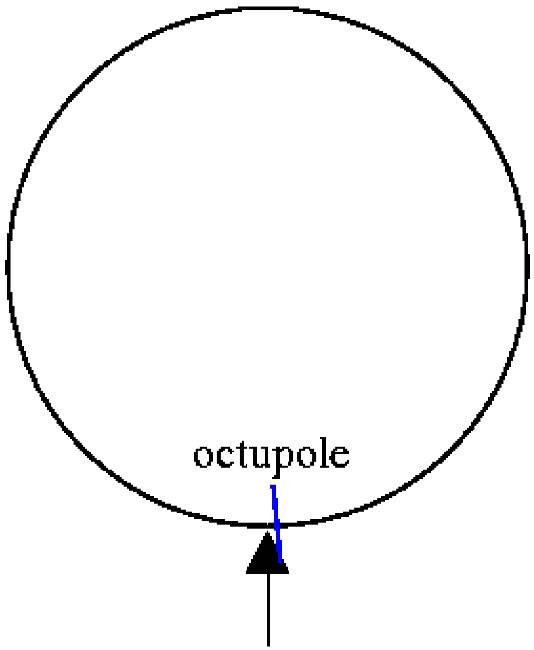
- **.... if Q_0 is close to K/N because then the**
  DIFFERENCE motion is SMALL !!



- **The N-turn Hamiltonian is a constant of the motion!**
- **Extracted particles follow contours, with a speed (step size) proportional to the local steepness**
- **Appropriate and useful for Slow Extraction Design & Analysis**

Octupole half-integer extraction

- **Similar to Tevatron slow extraction**



$$\mu = 2\pi\,(Q_0 - \frac{\text{odd integer}}{2}) \qquad (17)$$

- **Motion around first turn: kick & rotate**

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{0+} = \begin{pmatrix} x \\ x' - gx^3 \end{pmatrix}_0$$
$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 \approx \begin{pmatrix} -1 & -\mu \\ \mu & -1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{0+} \qquad (18)$$

- **Second turn is similar, for a net 2-turn motion of**

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 \approx \begin{pmatrix} 1 & 2\mu \\ -2\mu & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 - \begin{pmatrix} x \\ 2gx^3 \end{pmatrix}_0 \qquad (19)$$

- **Succinctly described by 2-turn Hamiltonian H_2**

$$H_2 = \frac{\mu}{2}(x^2 + x'^2) + \frac{g}{4}x^4 \tag{20}$$

- **Or, in action-angle co-ordinates**

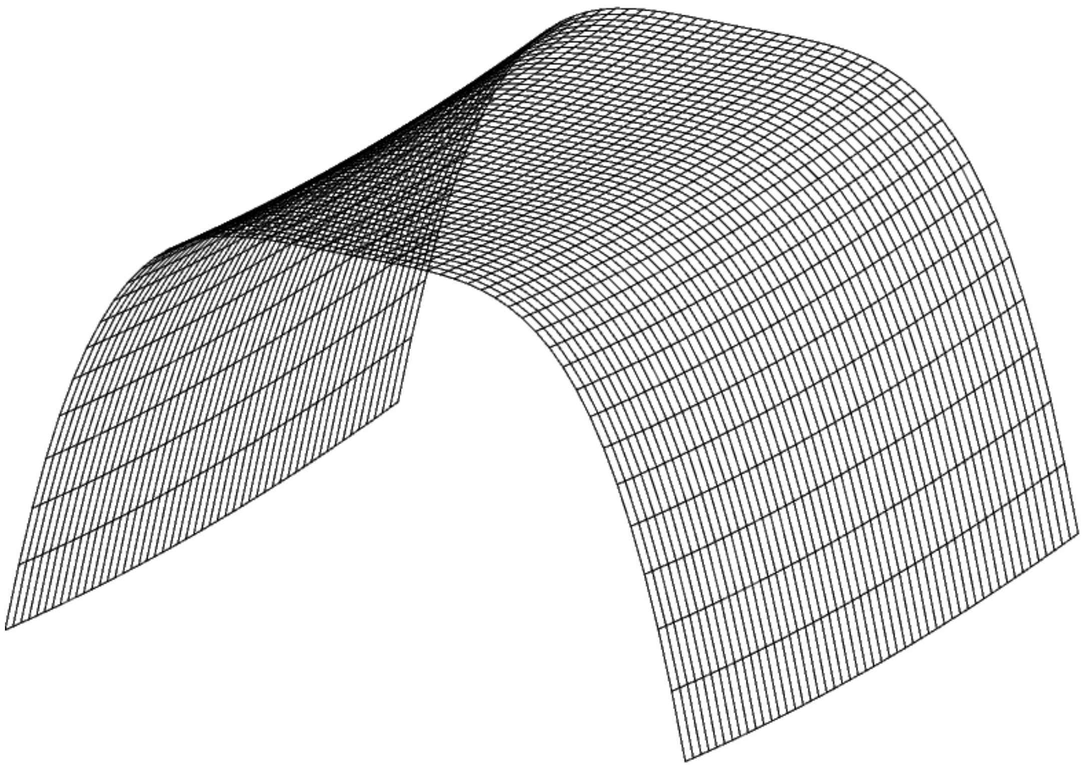
$$H_2 = \mu J + gJ^2 \sin^4(\phi) \tag{21}$$

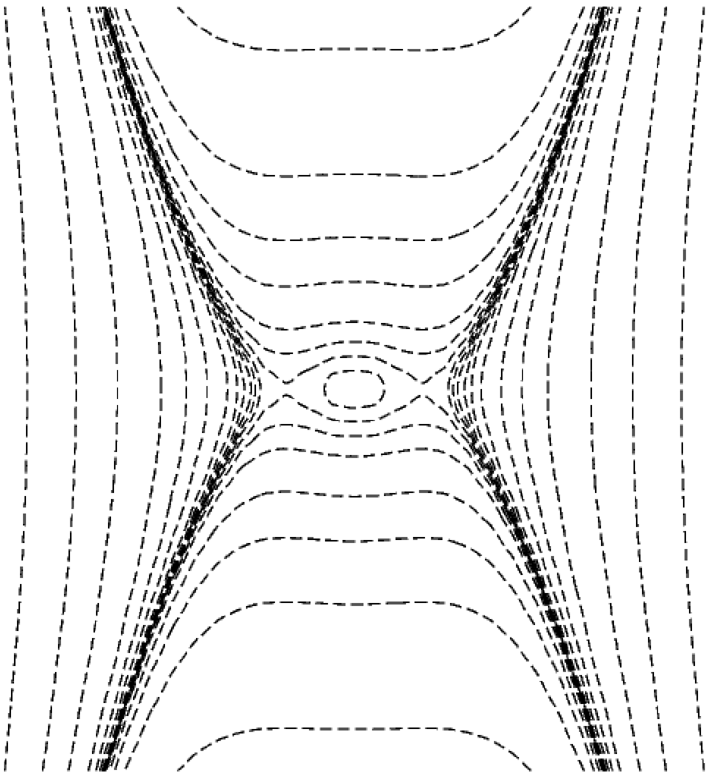
- **After expanding the trigonometric exponent for many octupoles**

$$H_2 = \mu J + [V_0 + V_2 \cos(2\phi + \phi_2) + V_4 \cos(4\phi + \phi_4)] J^2 \tag{22}$$

- **Complete parameterisation by $(V_0, V_2, V_4, \phi_2, \phi_4)$!**

- **For a single octupole, close to the ½ integer, get**





- **But, does it work? (see demo)**

Sextupole third-integer extraction

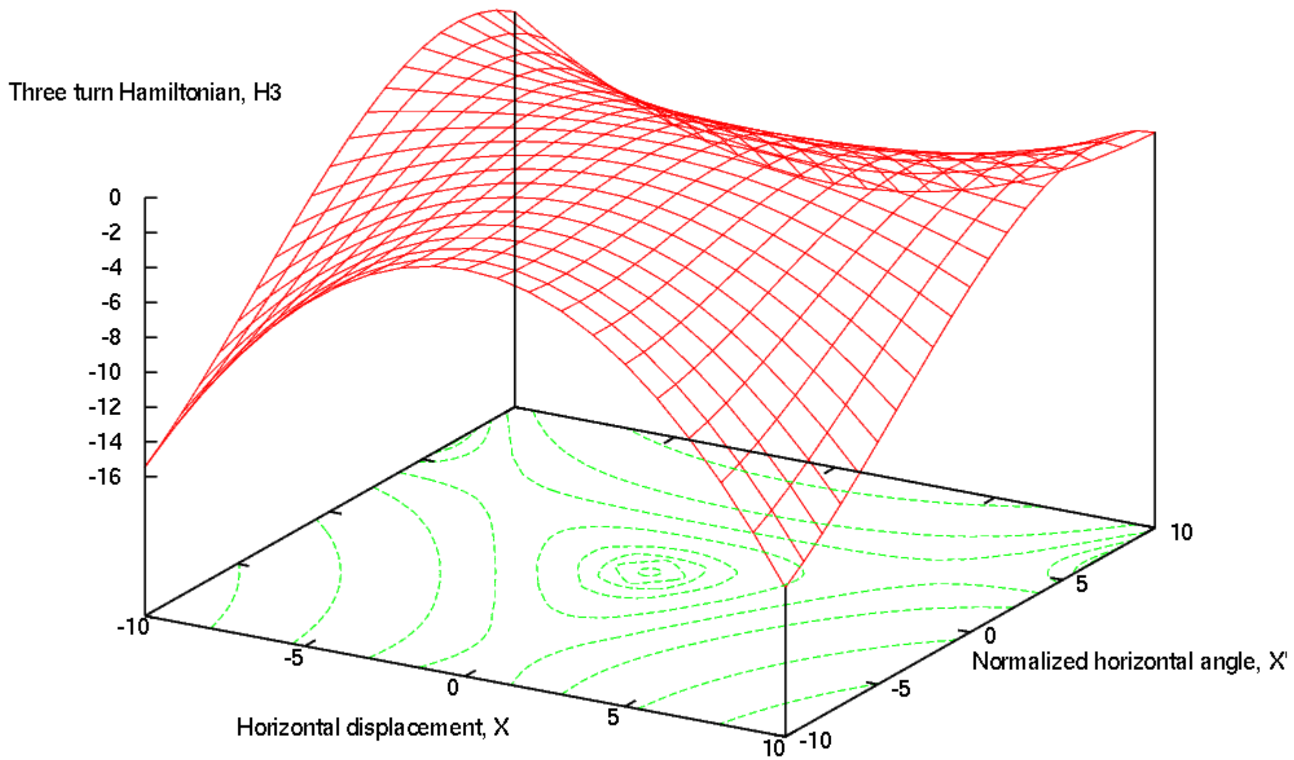
$$\mu \;=\; 2\pi\,(Q_0 \,-\, \frac{1}{3}) \tag{23}$$

- **After extending the previous analysis, and manipulating ...**

$$H_3 + \frac{g}{3}\left(\frac{-2\mu}{g}\right)^3 \;=\; \frac{g}{12}(x+\sqrt{3}x'+\frac{4\mu}{g})(x-\sqrt{3}x'+\frac{4\mu}{g})(x-\frac{2\mu}{g}) \tag{24}$$

- **... which becomes familiar when the RHS is zero - triangle!**

- **Like the octupole case, H_3 is first order in sextupole strength**



- **UNLIKE the octupole case, there is no explicit detuning**
- **How well does it work? (see demo)**

Summary

1) Traditional "continuous time" Hamiltonians are often outperformed by discrete time "Difference Hamiltonians".

2) One-turn Hamiltonians H_1 generate large steps, and are not constants of the motion, but connect to distortion functions and observable time series - eg, AC Dipole!

3) N-turn Hamiltonians H_N are constructed from H_1. They are conserved - follow the contours, measure the steepness!

4) Design H_2 or H_3 for half or third-integer slow extraction.

5) Watch out for applicability - eg, resonance islands, chaos!