DRAFT

Factor Gγ or (1+ Gγ) in Spin Dynamics

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The Froissart-Stora [1] formulation of the Thomas-BMT equation [2] may be written

$$\frac{d\vec{S}}{dt}=\frac{q}{m\gamma}\vec{S}\times[\vec{B}+G(\gamma\vec{B}_{\perp}+\vec{B}_{||})] \tag{1}$$

where $\vec{B}_{||}=(\hat{v}\cdot\vec{B})\hat{v}$ and $\vec{B}_{\perp}=\vec{B}-\vec{B}_{||}=(\hat{v}\times\vec{B})\times\hat{v}$ are the longitudinal and transverse parts of $\vec{B}$, $\hat{v}$ being the unit vector in the direction of the particle velocity. Similarly the longitudinal and transverse parts of the spin are
$\vec{S}_{||}=(\hat{v}\cdot\vec{S})\hat{v}$ and $\vec{S}_{\perp}=\vec{S}-\vec{S}_{||}=(\hat{v}\times\vec{S})\times\hat{v}$.

The Lorentz force equation is

$$\frac{d\hat{v}}{dt}=\frac{q}{m\gamma}\hat{v}\times\vec{B} \tag{2}$$

Combining (1) and (2) we obtain

$$\begin{aligned}\frac{d\vec{S}_{||}}{dt}&=\frac{q}{m\gamma}[(\frac{d\vec{S}}{dt}\cdot\hat{v}+\vec{S}\cdot\frac{d\hat{v}}{dt})\hat{v}+\hat{v}\cdot\vec{S}\frac{d\hat{v}}{dt}]\\&=\frac{q}{m\gamma}[\vec{S}_{||}\times\vec{B}_{\perp}+G\gamma(\vec{S}\times\vec{B}_{\perp}\cdot\hat{v})\hat{v}]\end{aligned} \tag{3}$$

and

$$\frac{d\vec{S}_{\perp}}{dt}=\frac{q}{m\gamma}[\vec{S}_{\perp}\times\vec{B}+G(\gamma\vec{B}_{\perp}+\vec{B}_{||})-G\gamma(\vec{S}\times\vec{B}_{\perp}\cdot\hat{v})\hat{v}] \tag{4}$$

The Froissart-Stora equation (1) is independent of the coordinate system. But, since the paricle moves in the vicinity of a closed orbit, it is convenient to use a coordinate system based on a closed *reference orbit* as we consider particles whose motion takes place near (though not exactly on) that orbit. We assume the reference orbit is plane and has a circumference we denote by $2\pi R$. We transform to a coordinate system (a Frenet-Serret system) based on this reference orbit. The position of a particle is characterized by the vector $\vec{\xi}$ from the point on the reference orbit closest to the particle, and we define the coordinates to be:

$s =$ the distance along the reference orbit from an origin point (arbitrarily chosen) on the reference orbit to the point on the reference orbit closest to the particle.

z = the vertical component of $\vec{\xi}$, i.e. the distance from the plane of the reference orbit to the particle.

x = the horizontal component of $\vec{\xi}$, which is the length of the projection of $\vec{\xi}$ on the orbit plane.

We also define $\rho(s)$ to be the radius of curvature of the reference orbit at s.; in a straight section the curvature $1/\rho(s)$ is zero, and the coordinates are locally Cartesian.

It is convenient to change to $\theta = s/R$ instead of the time t as the independent variable, with

$$d\theta = \frac{v/R}{1+x/\rho}dt \tag{5}$$

(note that s is the distance along the reference orbit, not exactly the distance traversed by the particle, and that $\theta = s/R$ is not exactly identical with the angle through which the particle has turned).

In what follows we shall sometimes use the prime for differentiation by θ; i.e.

$$X' \equiv \frac{dX}{d\theta}$$

for any variable X.

We define basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ to be the unit vectors in the x, s, z directions. These basis vectors rotate, with

$$\frac{d\hat{e}_1}{d\theta} = \frac{R\hat{e}_2}{\rho}; \quad \frac{d\hat{e}_2}{d\theta} = -\frac{R\hat{e}_1}{\rho}; \quad \frac{d\hat{e}_3}{d\theta} = 0 \tag{6}$$

Following Kondratenko [3] we introduce a "natural" or "local" reference frame based on the actual trajectory of the particle. The basis vector $\hat{u}_2$ is taken to be exactly the unit vector $\hat{v} = (\hat{e}_2 + x'\hat{e}_1/R + z'\hat{e}_3/R)/\sqrt{1+(x'^2+z'^2)/R^2}$ in the direction of the instantaneous particle velocity, and the other two are in the local radial and vertical direction orthogonal to $\hat{v}$:

$$\hat{u}_1 = [\hat{v}\times\hat{e}_3]_N \approx \hat{e}_1 - \frac{x'}{R}\hat{e}_2$$
$$\hat{u}_2 = \hat{v} \approx \hat{e}_2 + \frac{x'}{R}\hat{e}_1 + \frac{z'}{R}\hat{e}_3 \tag{7}$$
$$\hat{u}_3 = \hat{u}_1\times\hat{u}_2 \approx \hat{e}_3 - \frac{z'}{R}\hat{e}_2$$

where the subscript N denotes normalization to unit length, and the $\approx$ relations are correct to first order in the excursions x and z from the reference orbit.

The new basis vectors, of course, also rotate; using (6) and (7) we obtain, to first order in x and z,

$$\hat{u}_1' = \frac{R}{\rho}\hat{u}_2 - \frac{z'}{\rho}\hat{u}_3 - \frac{x''}{R}\hat{u}_2 = \left[\left(\frac{R}{\rho}-\frac{x''}{R}\right)\hat{u}_3 + \frac{z'}{\rho}\hat{u}_2\right]\times\hat{u}_1$$
$$\hat{u}_2' = -\frac{R}{\rho}\hat{u}_1 + \frac{z''}{R}\hat{u}_3 + \frac{x''}{R}\hat{u}_1 = \left[\left(\frac{R}{\rho}-\frac{x''}{R}\right)\hat{u}_3 + \frac{z''}{R}\hat{u}_1\right]\times\hat{u}_2 \tag{8}$$
$$\hat{u}_3' = \frac{z'}{\rho}\hat{u}_1 - \frac{z''}{R}\hat{u}_2 = \left(\frac{z'}{\rho}\hat{u}_2 + \frac{z''}{R}\hat{u}_1\right)\times\hat{u}_3$$

Here the independent variable θ and the excursions x and z are still defined with respect to the reference orbit, while the basis vectors are derived from the actual trajectory.

With θ as the independent variable the F-S equation is

$$\vec{S}' \equiv \frac{d\vec{S}}{d\theta} = \vec{S}\times\vec{F}; \quad \vec{F} = \frac{R}{B\rho}[\vec{B} + G(\gamma\vec{B}_\perp + \vec{B}_\parallel)] \tag{9}$$

where $B\rho = \dfrac{m\gamma v}{q}$ is the magnetic rigidity of the particle.

Using (8) we find for S_1, S_2 and S_3, the components of the spin along the "natural" basis vectors:

$$\begin{pmatrix} S_1' \\ S_2' \\ S_3' \end{pmatrix} = \begin{pmatrix} \vec{S}\times\vec{F}\cdot\hat{u}_1 + \vec{S}\cdot\hat{u}_1' \\ \vec{S}\times\vec{F}\cdot\hat{u}_2 + \vec{S}\cdot\hat{u}_2' \\ \vec{S}\times\vec{F}\cdot\hat{u}_3 + \vec{S}\cdot\hat{u}_3' \end{pmatrix} = \begin{pmatrix} \vec{S}\times\vec{W}\cdot\hat{u}_1 \\ \vec{S}\times\vec{W}\cdot\hat{u}_2 \\ \vec{S}\times\vec{W}\cdot\hat{u}_3 \end{pmatrix}$$

where (10)

$$\vec{W} = \vec{F} + \frac{z''}{R}\hat{u}_1 + \frac{z'}{\rho}\hat{u}_2 + \left(\frac{R}{\rho} - \frac{x''}{R}\right)\hat{u}_3$$

We may, following Courant and Ruth[4] and S Y Lee[5], express $\vec{B}_\perp$, $\vec{B}_\parallel$ and $\vec{F}$ in terms of the particle excursions, governed by the Lorentz force equation (2). In terms of the fixed vectors $\hat{e}$

$$\begin{aligned}\vec{B}_\perp &= B\rho\left(1-\frac{x}{\rho}\right)\left[\left(\frac{x''}{R^2}-\frac{1}{\rho}\right)\hat{e}_3+\frac{z'}{R\rho}\hat{e}_2-\frac{z''}{R^2}\hat{e}_1\right]\\ \vec{B}_\parallel &= \left[B_{sol}-\frac{B\rho}{R}\left(\frac{z}{\rho}\right)'\right]\hat{e}_2\end{aligned}\tag{11}$$

where we have added a solenoidal field B_{sol} (the longitudinal field on the reference orbit) which was not included in [4] and [5]). In the trajectory-based coordinate system (8) this becomes

$$\begin{aligned}\vec{B}_\perp &= B\rho\left(1-\frac{x}{\rho}\right)\left[\left(\frac{x''}{R^2}-\frac{1}{\rho}\right)\hat{u}_3-\frac{z''}{R^2}\hat{u}_1\right]\\ \vec{B}_\parallel &= \left[B_{sol}-\frac{B\rho}{R}\left(\frac{z}{\rho}\right)'\right]\hat{u}_2\end{aligned}\tag{12}$$

which is a slight simplification since $\vec{B}_\perp$ has no component in the direction $\hat{u}_2$. We thus have (to first order in the displacements x and z)

$$\begin{aligned}\vec{F} &= -(1+G\gamma)\frac{z''}{R}\hat{u}_1+(1+G)\left[\frac{RB_{sol}}{B\rho}-\left(\frac{z}{\rho}\right)'\right]\hat{u}_2\;+(1+G\gamma)\left(\frac{x''}{R}-\frac{R}{\rho}\right)\hat{u}_3\\ \vec{W} &= -G\gamma\frac{z''}{R}\hat{u}_1+(1+G)\left[\frac{RB_{sol}}{B\rho}-z\left(\frac{1}{\rho}\right)'\right]\hat{u}_2+G\gamma\left(\frac{x''}{R}-\frac{R}{\rho}\right)\hat{u}_3\end{aligned}\tag{13}$$

so that (10) becomes

$$\begin{aligned}S_1' &= -S_2G\gamma\left(\frac{R}{\rho}-\frac{x''}{R}\right)-S_3(1+G)\left[\frac{RB_{sol}}{B\rho}-z\left(\frac{1}{\rho}\right)'\right]\\ S_2' &= G\gamma\left[-S_3\frac{z''}{R}+S_1\left(\frac{R}{\rho}-\frac{x''}{R}\right)\right]\\ S_3' &= S_1(1+G)\left[\frac{RB_{sol}}{B\rho}-z\left(\frac{1}{\rho}\right)'\right]+S_2G\gamma\frac{z''}{R}\end{aligned}\tag{14}$$

Note that if G=0, i.e. if there is no anomalous magnetic moment, the longitudinal spin component S_2 is constant: helicity is conserved.

The dominant terms in the equations for S_1' and S_2' are $\mp G\gamma R/\rho$, leading to the precession frequency (spin tune) $G\gamma$. of oscillations in x and z.

The dominant depolarizing term (contribution to S_3') is the last term in the equation for S_3' and is proportional to $G\gamma$, not to (1+ $G\gamma$), in agreement with Kondratenko, Sivers and others [3]. In the calculation of depolarization due to transverse field perturbations (including magnet errors and rf excitation dipoles) and/or vertical betatron oscillations, appearing in much of the literature on spin dynamics including [4], [5] and [6] the basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are used; the relations corresponding to (14) in S Y Lee's book [5], rewritten in our notation, are

$$\begin{aligned}
S_x' &= -\left[G\gamma-(1+G\gamma)\frac{x''}{R}\right]S_s-\left[(1+G\gamma)\frac{z'}{\rho}-(1+G)\left(\frac{z}{\rho}\right)'\right]S_z \\
S_s' &= -\frac{z''}{R}(1+G\gamma)S_z+\left[G\gamma-(1+G\gamma)\frac{x''}{R}\right]S_x \\
S_z' &= \left[(1+G\gamma)\frac{z'}{\rho}-(1+G)\left(\frac{z}{\rho}\right)'\right]S_x+\frac{z''}{R}(1+G\gamma)S_s \\
&= \left[(\gamma-1)G\frac{z'}{\rho}-(1+G)z\left(\frac{1}{\rho}\right)'\right]S_x+\frac{z''}{R}(1+G\gamma)S_s
\end{aligned} \tag{15}$$

Conclusion

Both formulations (14) and (15) are correct. But the components S_1, S_2, S_3 addressed here in (14) have a direct physical significance, S_2 being the helicity (spin component along the velocity direction) which is strictly longitudinal, while S_1 and S_3 are strictly transverse components. The components S_x, S_s, S_z in (15), along the axes of the coordinate system defined by the reference orbit, all contain a mixture of the longitudinal and the transverse, and therefore have much less physical significance. Therefore (14) and not (15) is the relation that must be used in calculations of polarization, including resonance strength and strengths of (full or partial) Siberian snakes. Fortunately this makes very little practical difference because we almost always deal with large values of $G\gamma$. But in the case of deuterons G is small and negative, and indeed analysis of some recent COSY data, by Leonova and others, also point to the factor $G\gamma$, not $1+G\gamma$.

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[3] A M Kondratenko, contribution to Michigan-COSY teleconference 6/21/07; D W Sivers et al, contributions to Michigan-COSY teleconferences 3/20/07 and 4/19/07

[4] E D Courant and R D Ruth, BNL Report BNL-51270 (1980)

[5] S Y Lee, *Spin Dynamics and Snakes in Synchrotrons* (World Scientific, 1997), Chapters 2,3

[6] A W Chao and M Tigner (editors), *Handbook of Accelerator Physics and Engineering* (World Scientific, 1998), Sections 2.7.4, 2.7.5.