New Results for Spin Dynamics in Storage Rings

S.R. Mane

BNL, Nov 17, 2002

- New nonperturbative formalism MILES
  - Presented at Spin2002, BNL, Sept 2002
  - Exact analytical solution $\boldsymbol{n}$, ν single resonance model with 1, 2,... Snakes
- Summarize MILES
- Solution 2 Snakes: sine-factorials, sine-Bessels
- Results:
  - Spin Tune $\nu = 1/2$ off-axis $\sim$ K. Yokoya 1989
  - Snake resonances $\sim$ S. Y. Lee & S. Tepikian 1986
  - Maximum Tolerable Resonance Strength $\sim$ S. Y. Lee 1988
- Barber Spin 2002 $\sim$ Snake resonances not "true" resonances
- Fit Barber strob avg pol scan
- Nonresonant depolarization: zeroes of sine-Bessel
- New math results sine-Bessel: complete elliptic integral 1^{st} kind
- Measures of long-term polarization: $P_{\rm abs}$ & $P_{\rm fm}$
- $\boldsymbol{n}$ on Snake resonance line: "irreducibly discontinuous"

- New <u>nonperturbative</u> formalism MILES

  based exclusively on field-theoretic transformation of vector field

- Presented at Spin2002, BNL, Sept 2002

- <u>Exact</u> analytical solution $\boldsymbol{n}$, ν single resonance model with 1, 2,... Snakes

- Other models ...

- Miles

$$\boldsymbol{\sigma} \cdot \boldsymbol{n}(z_f) = M\, \boldsymbol{\sigma} \cdot \boldsymbol{n}(z_i)\, M^{-1}$$

$$M = \begin{pmatrix} f & -g^* \\ g & f^* \end{pmatrix}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \begin{pmatrix} n_3 & n_- \\ n_+ & -n_3 \end{pmatrix}$$

$$\begin{aligned} n_3(\boldsymbol{\phi}_* + \boldsymbol{\mu}) &= (ff^* - gg^*)n_3(\boldsymbol{\phi}_*) - f^*g^*n_+(\boldsymbol{\phi}_*) - fgn_-(\boldsymbol{\phi}_*) \\ n_+(\boldsymbol{\phi}_* + \boldsymbol{\mu}) &= 2f^*gn_3(\boldsymbol{\phi}_*) + f^{*2}n_+(\boldsymbol{\phi}_*) - g^2n_-(\boldsymbol{\phi}_*) \end{aligned}$$

- Fourier series:

$$n_3 = \sum_m n_{3m}\, e^{im\phi_*}$$

$$n_+ = \sum_m n_{+m}\, e^{im\phi_*}$$

• SRM with 2 Snakes

$$
\begin{aligned}
\boldsymbol{W} &= \nu_0\, \boldsymbol{e}_3 + \epsilon\, (\boldsymbol{e}_1 \,\cos\phi + \boldsymbol{e}_2 \,\sin\phi) + \text{Snakes} \\
\Omega &= \sqrt{(\nu_0 - Q)^2 + \epsilon^2} \\
\eta &= \frac{\epsilon}{\Omega}\,\sin(\,\frac{\pi\Omega}{2}\,) \\
\delta &= [Q] - \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
n_3 &= (1 - \eta^2)\, a \\
n_+ &= i g\, b \\
a &= a_0 + 2 \sum_{m=\text{even}} a_m \cos[m(\phi_* - \xi)] \\
b &= 2 \sum_{m=\text{odd}} b_m \sin[m(\phi_* - \xi)]
\end{aligned}
$$

- **Sine-Factorial**

$$\delta \equiv [Q] - \frac{1}{2}$$

$$S_m(\delta) = \sin(\pi\delta)\sin(2\pi\delta)\cdots\sin(m\pi\delta)$$

$$C_m(\delta) = \cos(\pi\delta)\cos(2\pi\delta)\cdots\cos(m\pi\delta)$$

- **Sine-Bessel**

$$\begin{aligned} a_m &= \frac{1}{\cos(m\pi\delta/2)} \sum_{k=0}^{\infty} \frac{C^2_{m/2+k}}{S_k S_{m+k}} (-1)^k \, \eta^{m+2k} \\ b_m &= \qquad\qquad \sum_{k=0}^{\infty} \frac{C^2_{(m-1)/2+k}}{S_k S_{m+k}} (-1)^k \, (\eta \, e^{i\pi\delta/2})^{m+2k} \end{aligned}$$

- Bessel functions

$$J_m(\eta) = \sum_{k=0}^{\infty} \frac{1}{k!\,(m+k)!} (-1)^k \, (\frac{\eta}{2})^{m+2k}$$

- **Spin Tune**

  $\nu = 1/2$ <u>all</u> orbits $[Q] =$ irrational

- Yokoya (1988) conjecture $\nu = 1/2$

  – confirm

- **Snake resonances**

- Zeroes of small denominators in sine-Bessel

$$[Q] \;=\; \frac{1}{2}\,\frac{2k_1+1}{2k_2+1} \;=\; \frac{1}{2}\,\frac{\mathrm{odd}}{\mathrm{odd}}$$

- confirm Lee & Tepikian 1986

- use $\delta = [Q] - \frac{1}{2}$

$$\begin{aligned} \delta \;=\; & 0 \\ & \frac{1}{3}\;\;\frac{2}{3} \\ & \frac{1}{5}\;\;\frac{2}{5}\;\;\frac{3}{5}\;\;\frac{4}{5} \\ & \frac{1}{7}\;\;\frac{2}{7}\;\;\frac{3}{7}\;\;\frac{4}{7}\;\;\frac{5}{7}\;\;\frac{6}{7} \end{aligned}$$

- N pairs $\Longrightarrow \delta \to \delta/N$

- **Max Tolerable Resonance Strength**
- N Snake pairs

$$\eta = \frac{\epsilon}{\Omega} \sin(\frac{\pi\Omega}{2N})$$

$$|\eta|_{\max} = 1 \qquad \Longrightarrow \begin{cases} \frac{\epsilon}{\Omega} = 1 \\ \sin(\frac{\pi\epsilon_{\max}}{2N}) = 1 \end{cases}$$

$$\frac{\pi\epsilon_{\max}}{2N} = \frac{\pi}{2}$$

$$\Longrightarrow \epsilon_{\max} = N$$

- confirm S. Y. Lee 1988

- Barber Spin2002
  - Snake resonances not "true" resonances
    - (Talk Spin2002 $\sim$ deny existence of Snake resonances)
  - $\boldsymbol{n}$ "irreducibly discontinuous" at Snake resonance line
  - Polarization scan
    - Depolarization at Snake resonances
    - Nonresonant depolarization
  - <u>*No explanation*</u>

- My reply . . .
- Fit Barber strob avg pol scan
- Nonresonant depolarization: zeroes of sine-Bessel
- Measures of long-term polarization: $P_{\rm abs}$ & $P_{\rm fm}$
  - Neither is "max"
- New math results sine-Bessel:
  - complete elliptic integral 1^{st} kind
- Plot $\boldsymbol{n}$ on Snake resonance line: *smooth, continuous*

- **Long-Term Polarization**

$$\begin{aligned}
\boldsymbol{P} &= \langle\, \boldsymbol{s} \,\rangle \\
\boldsymbol{P}_\infty &= \langle \langle\, \boldsymbol{s} \cdot \boldsymbol{n} \,\rangle\, \boldsymbol{n} \,\rangle \\
|\, \langle\, \boldsymbol{s} \cdot \boldsymbol{n} \,\rangle \,| &\leq 1 \\
\boldsymbol{P}_{\text{abs}} &= |\, \langle\, \boldsymbol{n} \,\rangle \,|
\end{aligned}$$

- $\sim$ "maximum attainable polarization"

- Alternative measure:

$$\begin{aligned}
\boldsymbol{s} &= \boldsymbol{n}_0 \\
\boldsymbol{P}'_\infty &= \langle\, \boldsymbol{n} \cdot \boldsymbol{n_0} \;\; \boldsymbol{n} \,\rangle \\
\boldsymbol{P}_{\text{fm}} &= \boldsymbol{P}'_\infty \cdot \boldsymbol{n}_0 \\
&= \langle\, (\boldsymbol{n} \cdot \boldsymbol{n}_0)^2 \,\rangle
\end{aligned}$$

- Plot both P_{abs} and P_{fm}

- SRM with 2 Snakes

$$\langle\, n_3 \,\rangle \;=\; (1-\eta^2)\, a_0$$

$$P_{\rm abs} \;=\; (1-\eta^2)\, |\, a_0 \,|$$

$$P_{\rm fm} \;=\; (1-\eta^2)^2\; \Big\{ a_0^2 + 2\,(\, a_2^2 + a_4^2 + a_6^2 + \cdots\,)\Big\}$$

- Sine-Bessel — New Results
- $[Q]$ = rational, not on Snake resonance

$$a_0 \;=\; A_0(\eta, \delta)\, K(\eta^4)$$

- A_0 = polynomial
- complete elliptic integral of 1^{st} kind

$$K(k) \;=\; \int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2 \sin^2 u}}$$

- low orders:

$$\begin{aligned} a_2 \;&=\; A_0(\eta, \delta)\, K(\eta^4) \\ b_1 \;&=\; B_1(\eta, \delta)\, K(\eta^4) \\ b_3 \;&=\; B_3(\eta, \delta)\, K(\eta^4) \end{aligned}$$

- higher orders: derivatives of elliptic integral
  - Future work

- Cute result — control nonresonant depolarization
- nonresonant depolarization:

$$a_0 = 0$$
$$A_0 = 0$$

- Special case $[Q] = 1/3$:

$$A_0 = 1 - 3\eta^2 + \eta^4$$

- Equate to zero:

$$\eta = \frac{\sqrt{5} - 1}{2}$$

- inverse of Golden Ratio