Modeling the emittance evolution from the IPM measurements at the Fermilab Booster

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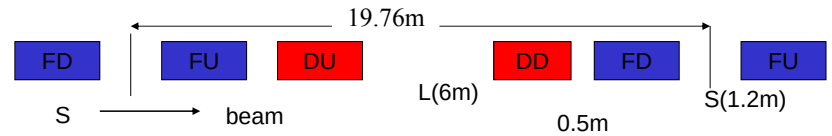
Since 1996, the Fermilab Booster is equipped with turn-by-turn beam profile measurements. This is the first attempt to systematically measure the evolution of the beam emittances during the acceleration cycle, and their dependence on the beam intensity. This effort is important because the demand of neutrino-experiments: The proton demand on the Booster has increased from 7E15 to as much as 1.8E17 protons/hr because of the MiniBooNE and NuMI projects. [see X. Huang, S.Y. Lee, K.Y. Ng, and Y. Su, PRSTAB **8**, 014202 (2006)]

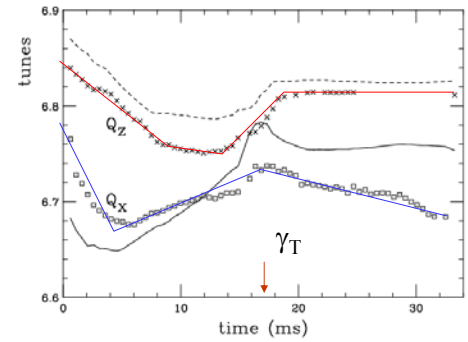
1. Introduction
2. Measurements and Modeling in emittance reduction
3. Modeling by numerical simulations
4. Implications on the FFAG accelerators
5. Conclusions

An Overview of Booster's lattice



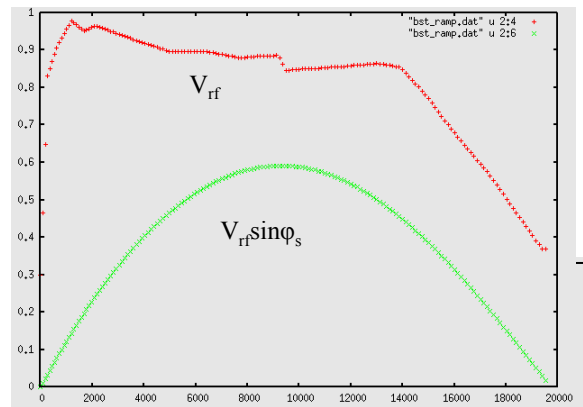
$\beta_{x,\max} = 33.7\,\mathrm{m}$ $\beta_{z,\max} = 20.4\mathrm{m}$ $D_{\max} = 3.2\mathrm{m}$

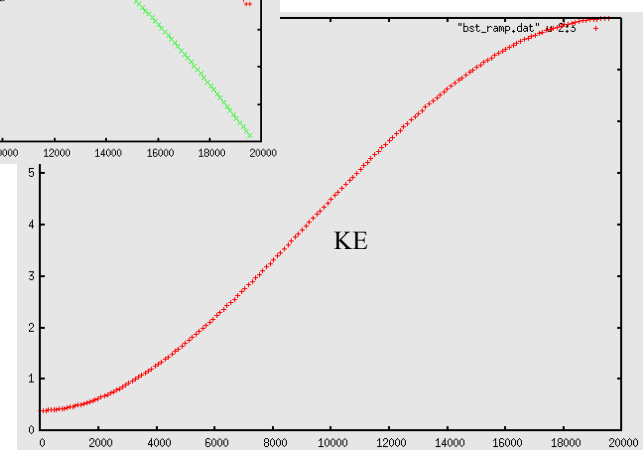
- Superperiod=24
- In each period, four Combined-function magnets. Length 2.889 m. Gradients are $K_1(F)=0.0542\ m^{-2}$ and $K_1(D)=-0.0577\ m^{-2}$. The integrated focusing strengths are $0.313\ m^{-1}$ and $-0.333\ m^{-1}$.
- Horizontal tune 6.7, vertical tune 6.8
- Extraction doglegs in L03 and L13 perturb the lattice functions



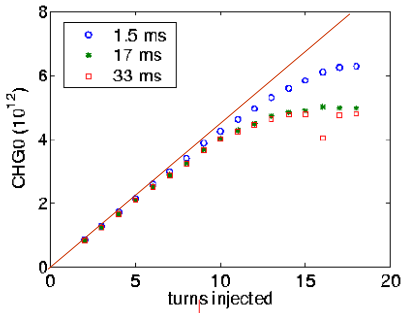
The Fermilab Booster

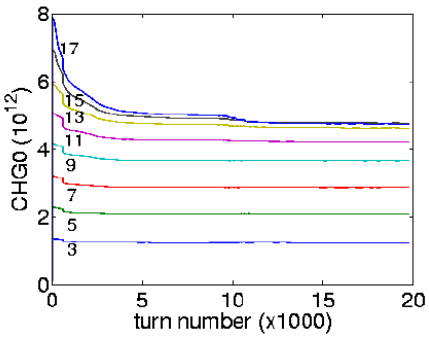
Circumference(m)	474.2
Inj/ext energy (GeV)	0.4/8.0
Cycling rate (Hz)	15
Hori/vert betatron tunes	6.7/6.8
Superperiod	24
Transition gamma	5.446
Harmonic number	84
Protons per pulse	5.0E12
Longitudinal 95% emittance (eV-s)	0.10 at injection
Transverse 95% emittance (mm-mr)	12π (normalized) at injection

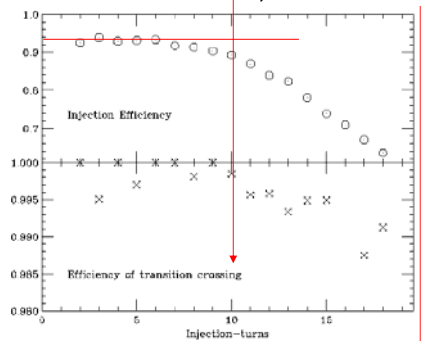




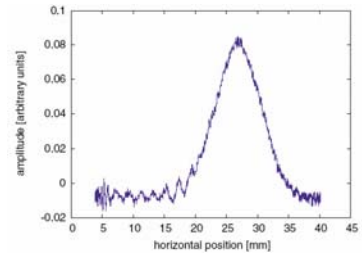
Typical Injection scheme at the Fermilab Booster: H- at 400 MeV from Linac is about 30 mA. Strip injection into the Booster gives about 4.2×10^{11} protons per injection turn. At 400 MeV, the revolution frequency is 4.51×10^{5} Hz. The Normal operation has 12 injection turns, and the total intensity is about 5×10^{12} ppp at 15 Hz rep-rate. In fact, when the injection turn is larger than 12, the Booster loss problem becomes severe.







Ionization beam profile monitors (IPM's) measure the beam profile by collecting electrons from background gas ionization. Measurement accuracy will depend on the number of electrons collected. The number of electrons collected depends on pressure, bunch intensity, and data collection mode. The center-to-center distance between microchannels is of the order of 1.5mm. Thus one can measure the beam width to within 1.5-2.0 mm. For the IPM at Fermilab Booster, the space charge correction has been shown to be important [J. Amundsen et al PRSTAB **6** 102801 (2003)]



$$\varepsilon = \frac{\sigma^2}{\beta}$$

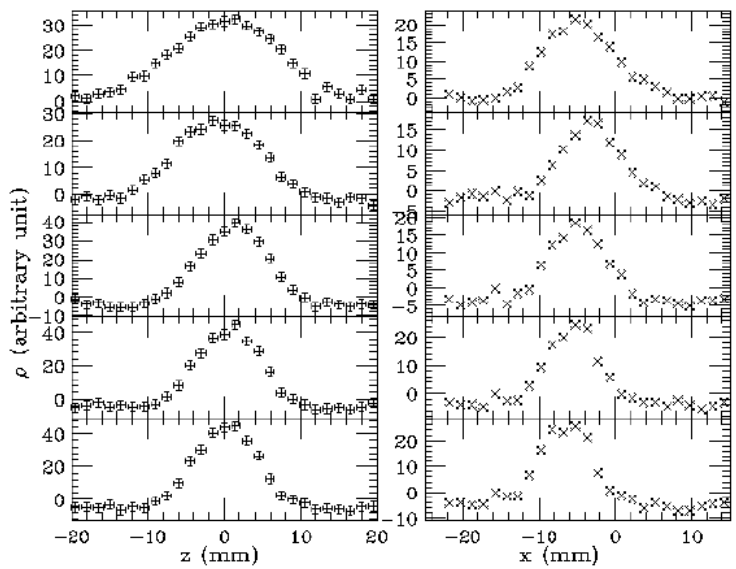
500-rev

4000-rev

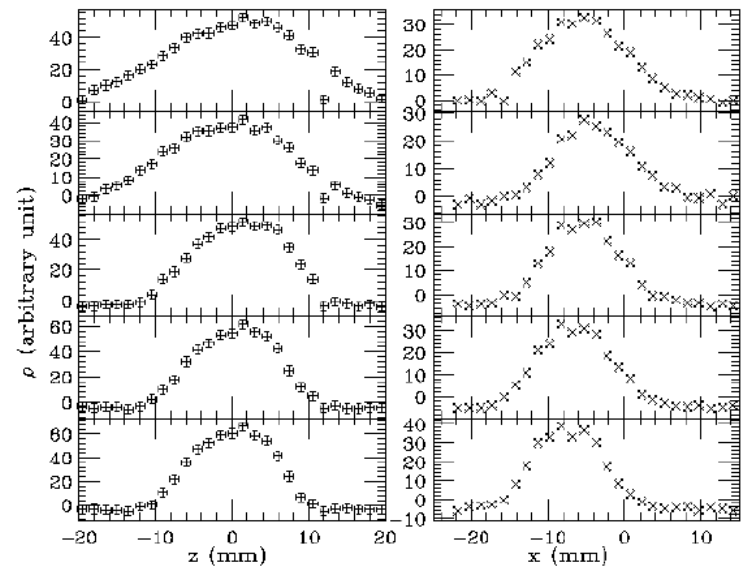
8000-rev

12000-rev

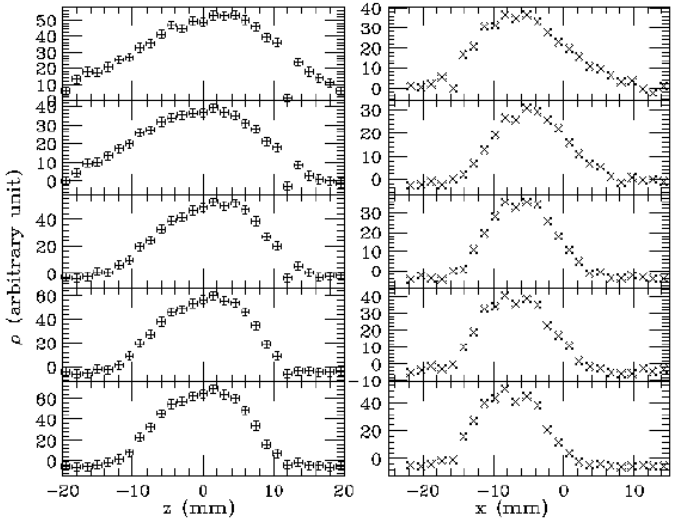
16000-rev



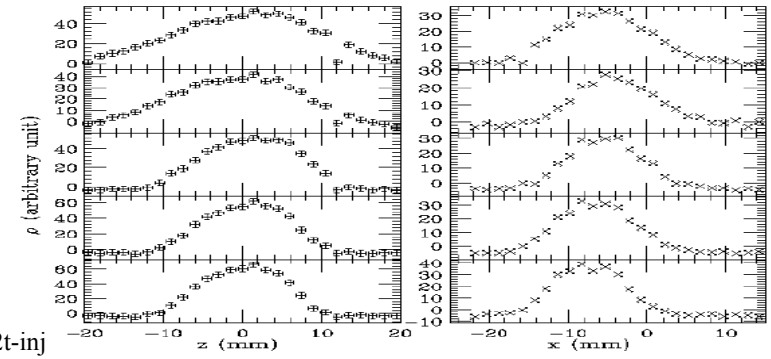
6t-inj



12t-inj



14t-inj



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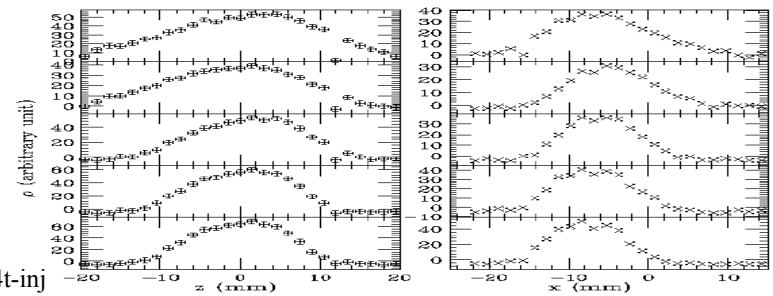

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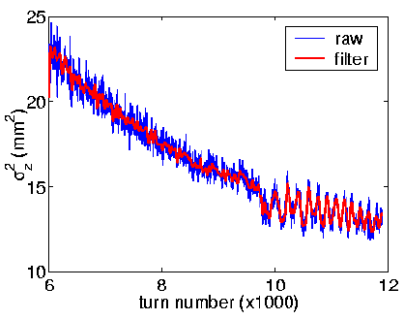
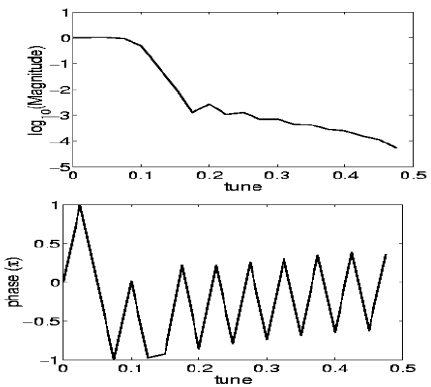
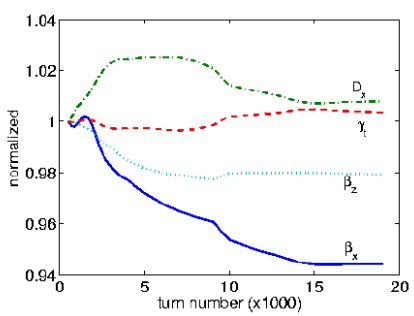
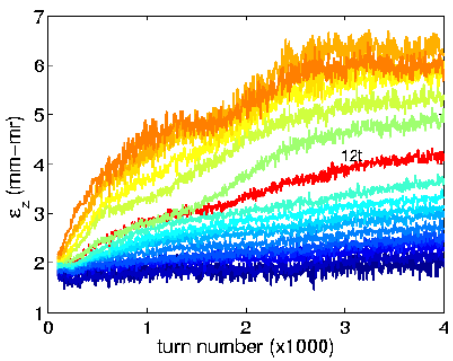


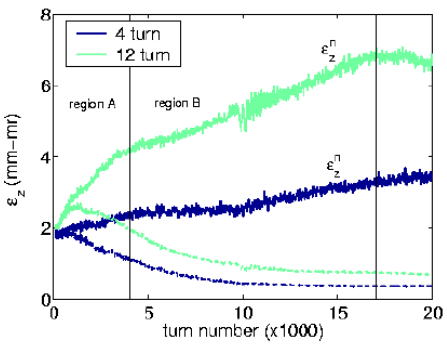




Machine modeling:
βx=6.3, βz=21.4, Dx=2.54,
at the IPM location with
Qx=6.7, Qz=6.8

Space charge effects: Linac delivers about 30 mA beam current to the Fermilab Booster, i.e. about 4.2×10^{11} particles in one injection turn.





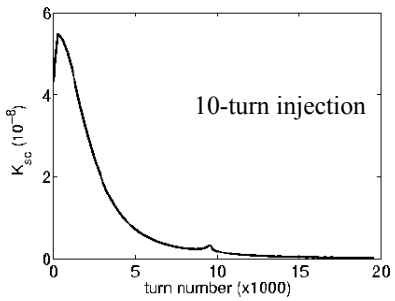
$$\varepsilon = a_0 + b_1 t + b_2 \int_0^t K_{sc}\, dt \qquad K_{sc} = \frac{2Nr_0}{\beta^2\gamma^3}$$

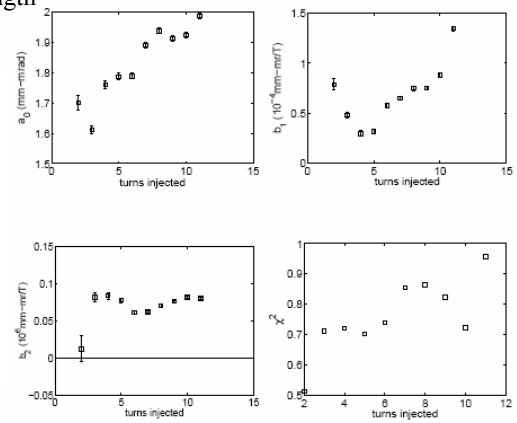
The space charge perveance can be calculated from the beam parameters, and there are three fitting parameters, a_0, b_1, b_2.

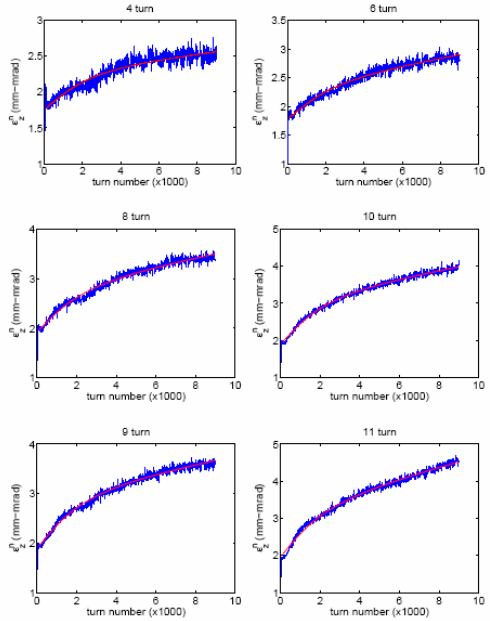
$$K_{sc} = \frac{2Nr_0}{\beta^2\gamma^3}$$

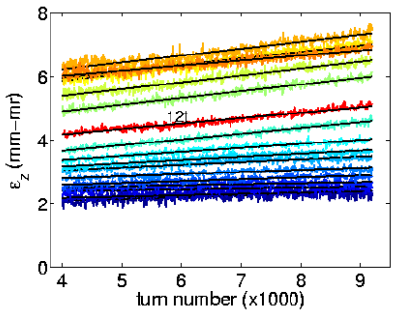
$$\varepsilon = a_0 + b_1 t + b_2 \int_0^t K_{sc}\, dt$$

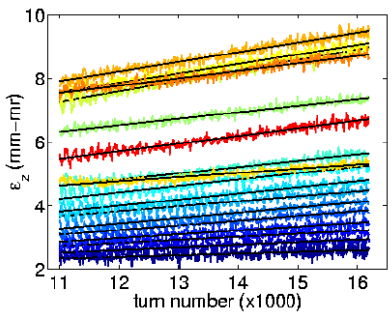
N= number of particles per unit length

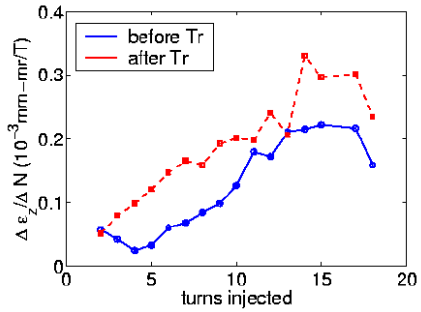








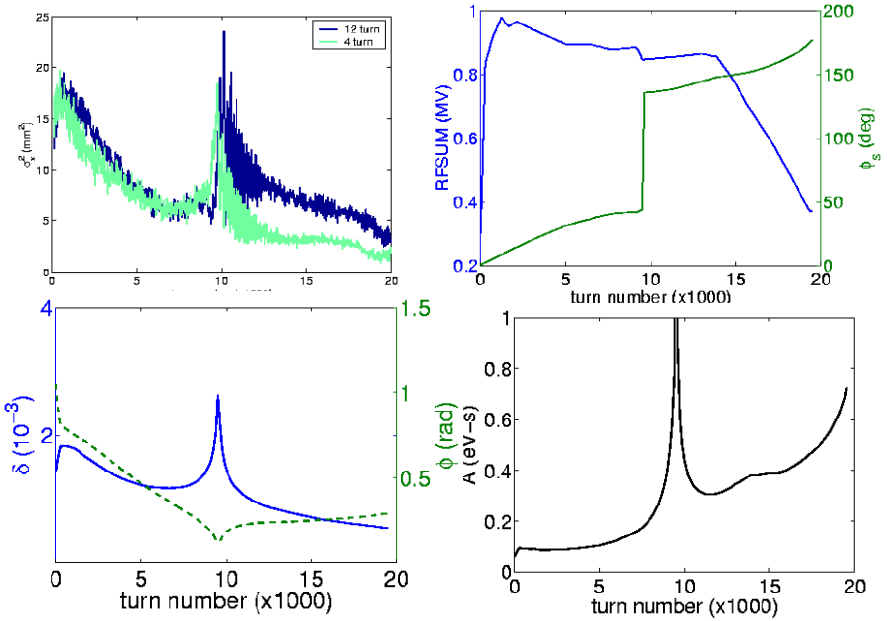




Summary:

1. $d\varepsilon_z/dt \sim b + K_{sc}$
2. ε_z increases linearly with t.

Horizontal IPM measurements

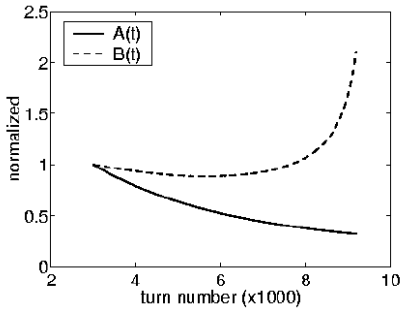


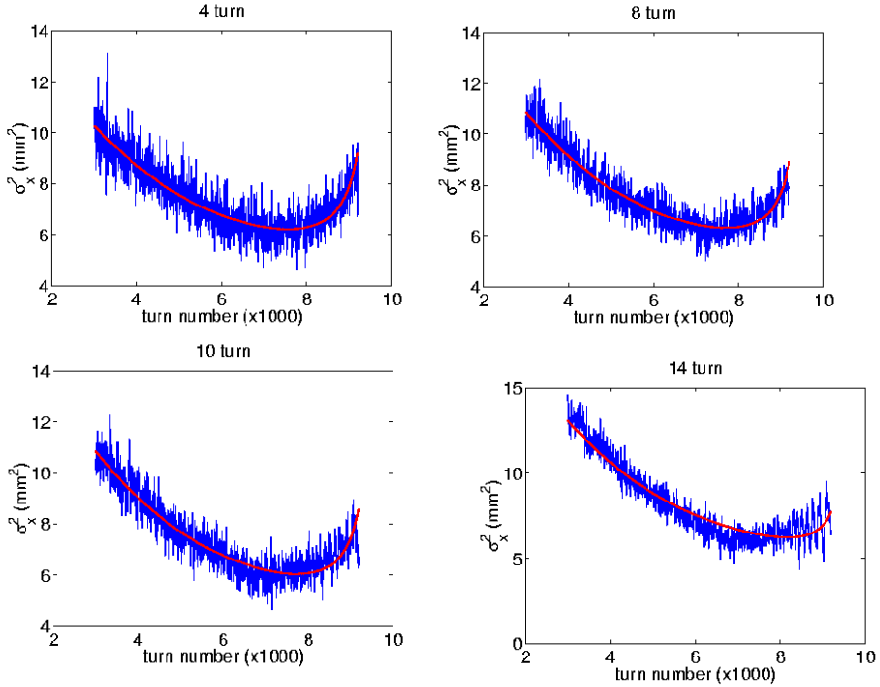
$$\sigma_x^2 = \beta_x \epsilon_{\rm rms} + D^2 \sigma_\delta^2 = \beta_x \frac{\epsilon^n_{\rm rms}}{\beta\gamma} \frac{\beta_{x0}(\beta\gamma)_0}{\beta_{x0}(\beta\gamma)_0} + D^2 \sigma_{\delta 0}^2 \frac{\sigma_\delta^2}{\sigma_{\delta 0}^2} = aA(t) + bB(t)$$

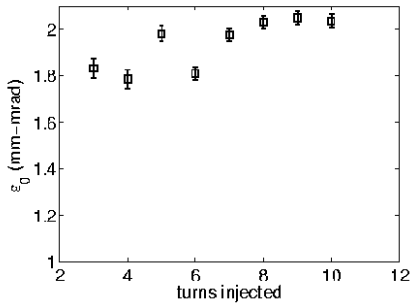
$$a = \epsilon^n_{\rm rms} \frac{\beta_{x0}}{\beta_0 \gamma_0}, \quad A(t) = \frac{\beta_x}{\beta_{x0}} \frac{\beta_0 \gamma_0}{\beta\gamma}$$

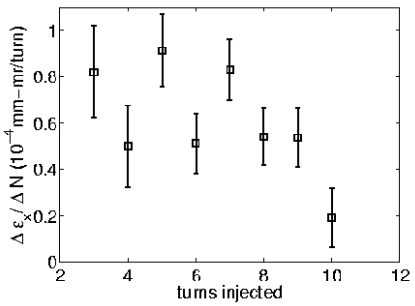
$$b = D^2 \sigma_{\delta 0}^2, \quad B(t) = \frac{\gamma_0 \sqrt{\gamma_0 |\eta_0| / V_0} |\cos \phi_{s0}|}{\gamma \sqrt{\gamma |\eta| / V} |\cos \phi_s|},$$

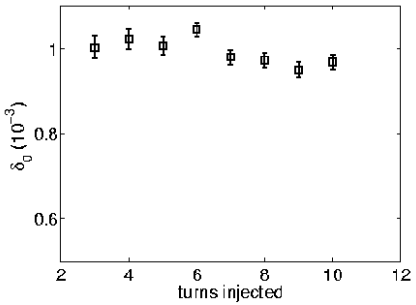
$$\sigma_x^2 = (a_0 + a_1 t) A(t) + b_0 B(t).$$

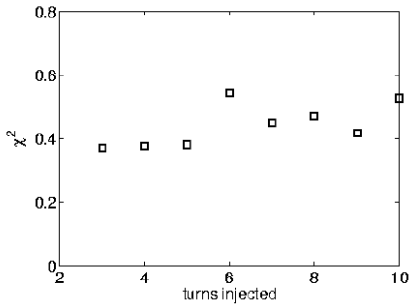


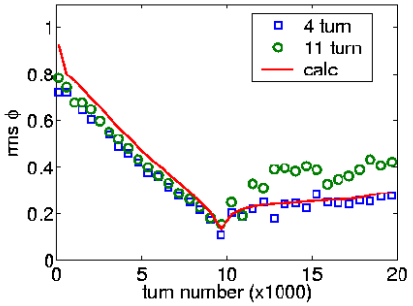


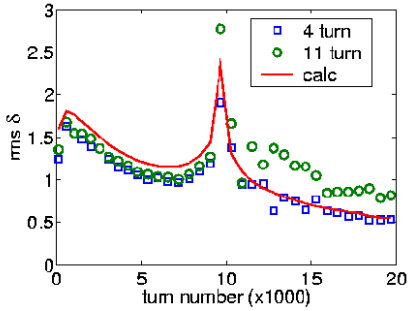


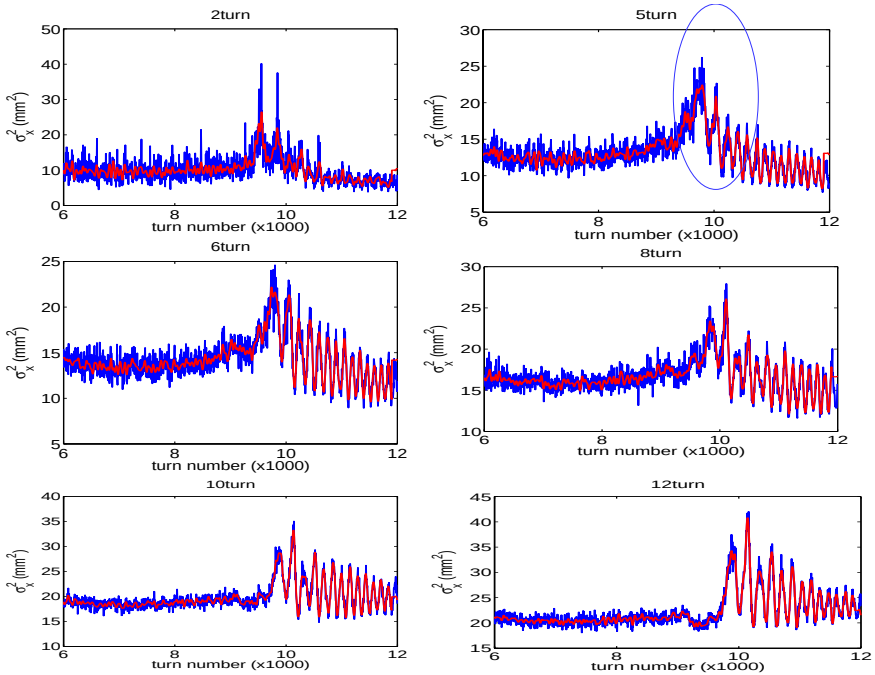












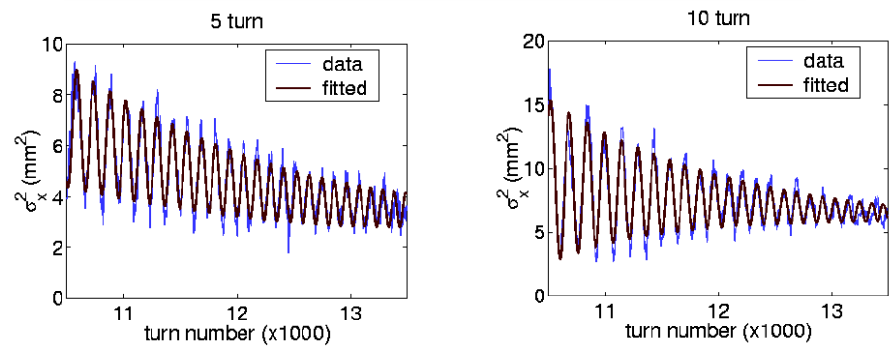
Post Transition Horizontal beam size oscillation

$$\sigma_x^2(t) = a + bt + ct^2 + A\exp(-\alpha t)\cos[2\pi(f_1 t + f_2 t^2) + \chi],$$

$$2A = \sigma_{x,\max}^2 - \sigma_{x,\min}^2.$$

$$\sigma_{x,\text{static}}^2 = a + bt + ct^2$$

$$\sigma_{x,\text{static}}^2 = \beta_x \epsilon_{x,\text{rms}} + D^2\bar{\delta}^2 = \frac{\beta_x}{\beta\gamma}\epsilon_{x,\text{rms}}^n + D^2\bar{\delta}^2 = \frac{\beta_x}{\beta\gamma}\epsilon_{x0}^n(1+\alpha_x t) + D^2\bar{\delta}^2,$$



$$\sigma_x^2 = \beta_x \varepsilon_{\text{rms}} + D^2\delta^2$$
$$= a + bt + ct^2 + Ae^{-\alpha t}\cos[2\pi(f_1 t + f_2 t^2) + \varphi]$$

$$\delta = (\Delta p/p)_{\text{rms}}$$

$$a + bt + ct^2 = \beta_x \varepsilon_{\text{rms}} + D^2\bar{\delta}^2$$

$$f_{\text{syn}}(t) = f_1 + 2f_2 t$$

$$2A = D^2(\delta_1^2 - \delta_2^2)$$

$$\delta_1 = \frac{\gamma_T}{3^{1/6}\beta\tau_{\text{ad}}\Gamma(\frac{2}{3})}\left(\frac{2A_{t\Delta E}}{3mc^2\dot{\gamma}}\right)^{1/2}$$

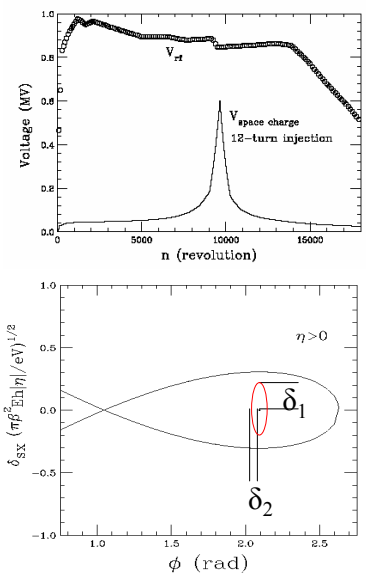
$$\delta_2 = \frac{\nu_s A_{\phi\delta}}{h\eta\pi\delta_1}$$

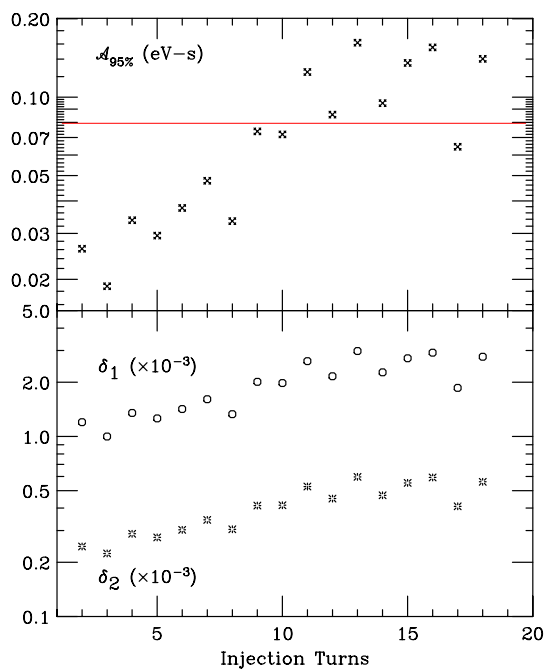
$$A_{\phi\delta} = \frac{h\omega_0}{\beta^2\gamma_T mc^2}A_{t\Delta E}$$

$$\tau_{\text{ad}} = \left(\frac{\pi\beta^2\gamma_T^4 mc^2}{\dot{\gamma}\omega_0^2 heV|\cos\phi_s|}\right)^{1/3}$$

$$\tau_{\text{ad}} = \left(\frac{\pi\beta^2 mc^2\gamma_T^4}{\dot{\gamma}\omega_0^2 heV|\cos\phi_s|}\right)^{1/3} \approx 0.20 \text{ ms},$$

$$\tau_{\text{nl}} = \gamma_T^3\frac{\eta_1}{2\dot{\gamma}}\hat{\delta} = \gamma_T\frac{\frac{3}{2}\beta_0^2 + \gamma_T^2\alpha_1}{2\dot{\gamma}}\hat{\delta} \approx 0.07 \text{ ms},$$





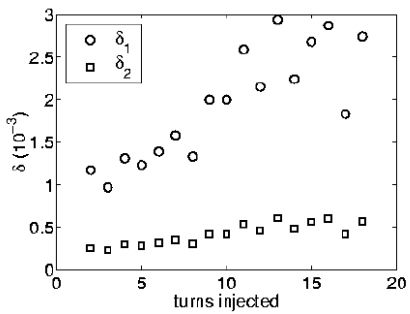
Bucket area after transition

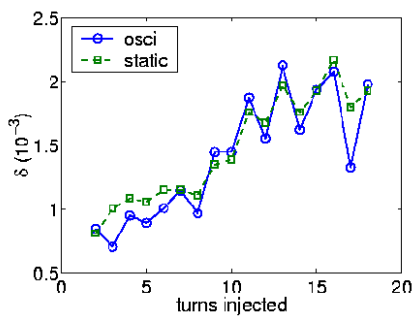
Bucket area at injection

$\delta_1/\delta_2 \sim 5$

$$\bar{\delta}_{\text{static}} = \frac{\sqrt{\sigma_x^2 - \beta_x \epsilon_{x,\text{rms}}}}{D},$$

$$\bar{\delta}_{\text{osci}} = \sqrt{\frac{\delta_1^2 + \delta_2^2}{2}},$$





Summary:

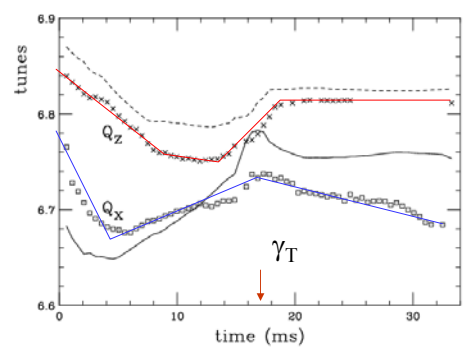
1. $d\varepsilon_z/dt \sim b+\mathbf{K_{sc}}$
2. ε_z increases linearly with t at about 1 π-mm-mrad in 10^4 revolutions.
3. The horizontal emittance and the off-momentum spread can be separated by using different scaling law (energy dependence).
4. The **horizontal emittance is less affected by the space charge force**! **Why?**
5. The slow linear growth of the horizontal emittance is the same as that of the vertical plane!
6. The post-transition horizontal bunch-width oscillation is induced essentially by the longitudinal mis-match of the bunch shape with space charge effect in the rf potential well. Using the bunch shaoe mis-match, one can deduce the phase space area.

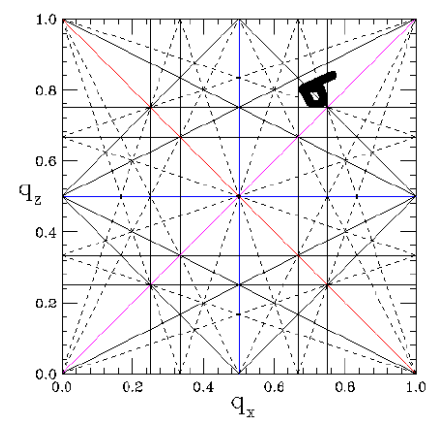
Modeling algorithm:

We consider N-particle in Gaussian distribution and construct a model with 24 Superperiod FODO cells

$$M_{\text{D}\to\text{F}} = \begin{pmatrix} \sqrt{\frac{\beta_{x,\text{F}}}{\beta_{x,\text{D}}}}\cos\psi_x & \sqrt{\beta_{x,\text{F}}\beta_{x,\text{D}}}\sin\psi_x & 0 & 0 \\ -\frac{1}{\sqrt{\beta_{x,\text{F}}\beta_{x,\text{D}}}}\sin\psi_x & \sqrt{\frac{\beta_{x,\text{D}}}{\beta_{x,\text{F}}}}\cos\psi_x & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\beta_{z,\text{F}}}{\beta_{z,\text{D}}}}\cos\psi_z & \sqrt{\beta_{z,\text{F}}\beta_{z,\text{D}}}\sin\psi_z \\ 0 & 0 & -\frac{1}{\sqrt{\beta_{z,\text{F}}\beta_{z,\text{D}}}}\sin\psi_z & \sqrt{\frac{\beta_{z,\text{D}}}{\beta_{z,\text{F}}}}\cos\psi_z \end{pmatrix}$$

$$M_{\text{F}\to\text{D}} = \begin{pmatrix} \sqrt{\frac{\beta_{x,\text{D}}}{\beta_{x,\text{F}}}}\cos\psi_x & \sqrt{\beta_{x,\text{F}}\beta_{x,\text{D}}}\sin\psi_x & 0 & 0 \\ -\frac{1}{\sqrt{\beta_{x,\text{F}}\beta_{x,\text{D}}}}\sin\psi_x & \sqrt{\frac{\beta_{x,\text{F}}}{\beta_{x,\text{D}}}}\cos\psi_x & 0 & 0 \\ 0 & 0 & \sqrt{\frac{\beta_{z,\text{D}}}{\beta_{z,\text{F}}}}\cos\psi_z & \sqrt{\beta_{z,\text{F}}\beta_{z,\text{D}}}\sin\psi_z \\ 0 & 0 & -\frac{1}{\sqrt{\beta_{z,\text{F}}\beta_{z,\text{D}}}}\sin\psi_z & \sqrt{\frac{\beta_{z,\text{F}}}{\beta_{z,\text{D}}}}\cos\psi_z \end{pmatrix}$$





Space charge force is a local kick on every half cell:

$$\rho(x,z) = \frac{Ne}{2\pi\sigma_x\sigma_z}\exp\{-\frac{x^2}{2\sigma_x^2}-\frac{z^2}{2\sigma_z^2}\},$$

$$\begin{aligned} V(x,z) &= \frac{Nr_0}{\beta^2\gamma^3}\int_0^\infty \frac{1-\exp\{-\frac{x^2}{2\sigma_x^2+t}-\frac{z^2}{2\sigma_z^2+t}\}}{\sqrt{(2\sigma_x^2+t)(2\sigma_z^2+t)}}dt \\ &\approx \frac{Nr_0}{\beta^2\gamma^3}\left(\frac{x^2}{\sigma_x(\sigma_x+\sigma_z)}+\frac{z^2}{\sigma_z(\sigma_x+\sigma_z)}\right) \\ &\quad -\frac{Nr_0}{4\beta^2\gamma^3\sigma_x^2(\sigma_x+\sigma_z)^2}\left(\frac{2+R}{3}x^4+\frac{2}{R}x^2z^2+\frac{1+2R}{3R^3}z^4\right)+\cdots \end{aligned}$$

$$\begin{aligned} \Delta x' &= -\frac{\partial V}{\partial x}\ell \approx \frac{2Nr_0\ell}{\beta^2\gamma^3\sigma_x(\sigma_x+\sigma_z)}x\exp\{-\frac{x^2+z^2}{(\sigma+\sigma_z)^2}\}, \\ \Delta z' &= -\frac{\partial V}{\partial z}\ell \approx \frac{2Nr_0\ell}{\beta^2\gamma^3\sigma_z(\sigma_x+\sigma_z)}z\exp\{-\frac{x^2+z^2}{(\sigma+\sigma_z)^2}\}, \end{aligned}$$

- Sextupole nonlinearity on each half cell for nonlinearity in dipoles
- Linear coupling,
- Random quadrupoles with zero tune shifts
- Random closed orbit error
- Dynamical aperture of 80 by 50 pi-mm-mrad

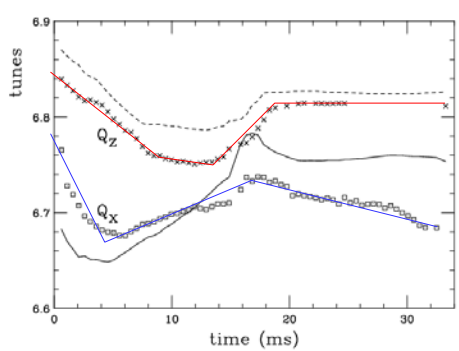
$$x'' + K_x(s)x = \frac{b_0(s)}{\rho} + \frac{b_1(s)}{\rho}x + \frac{a_1(s)}{\rho}z + \frac{1}{2}\frac{b_2(s)}{\rho}(x^2 - z^2)$$

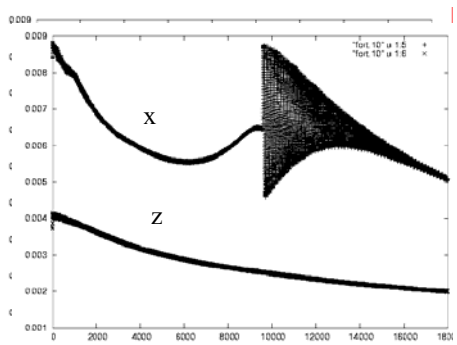
$$z'' + K_z(s)z = -\frac{a_0(s)}{\rho} - \frac{b_1(s)}{\rho}z + \frac{a_1(s)}{\rho}x - \frac{b_2(s)}{\rho}xz$$

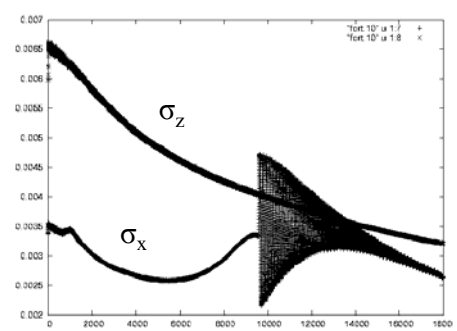
Random number generators are used to generate b0, a0, b1, and a1. The quadrupole error is subject to a constraint with zero tune shift. The integrated sextupole strengths are set to the systematic values: -0.0173 m^{-2} and -0.263 m^{-2} for focusing and defocusing dipoles respectively.

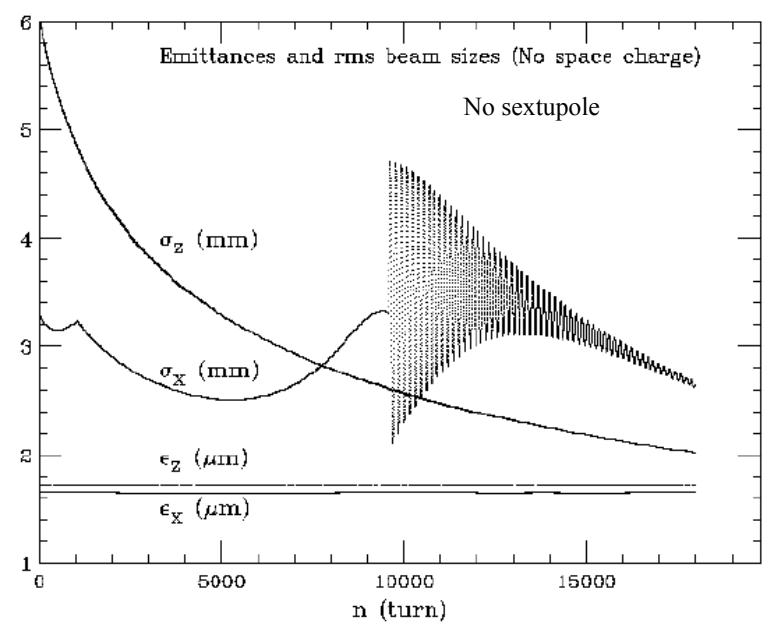
$$\begin{aligned} \delta_{\rm rms}(n) &= \delta_{\rm rms}(1)B_f(n)\,(1+(G_\delta-1)(1-\exp(-\alpha_g(n-n_t))) \\ &\quad [1+A_\delta\exp(-\alpha_g(n-n_t))\sin(2\pi(n-n_t)f)]\,, \end{aligned}$$

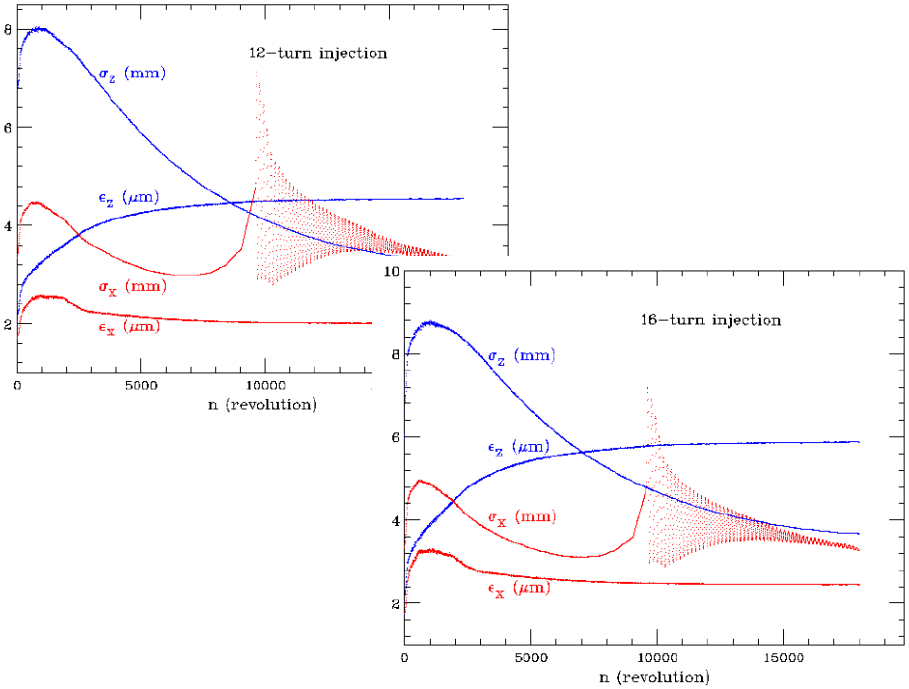
(For $n>n_t$=9600) G_δ=2, A_δ=0.5, f=1/150, α_δ=1/(15*150)

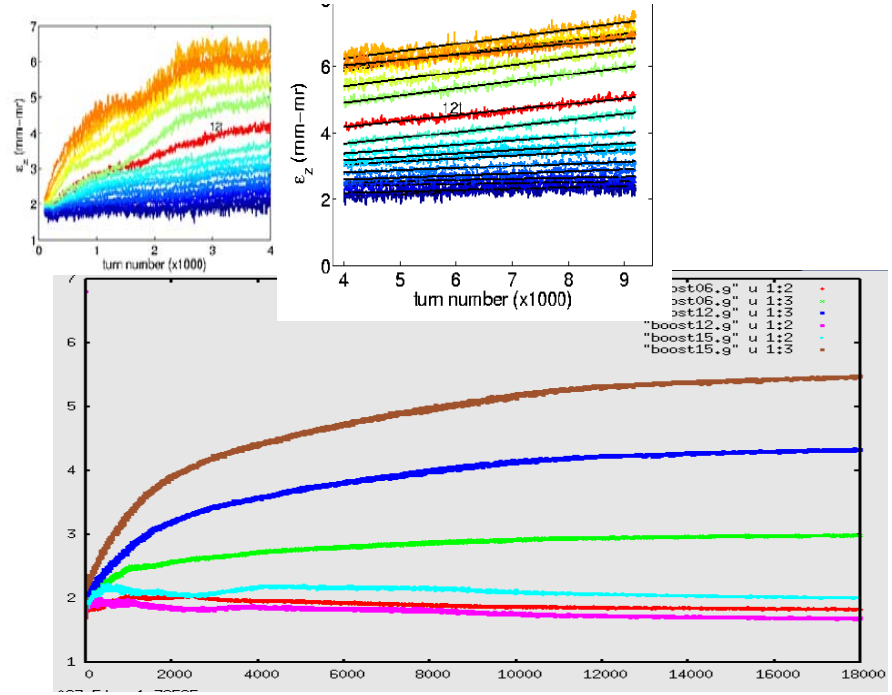






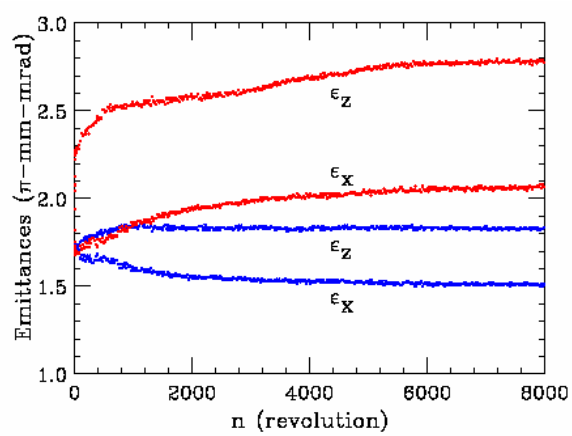




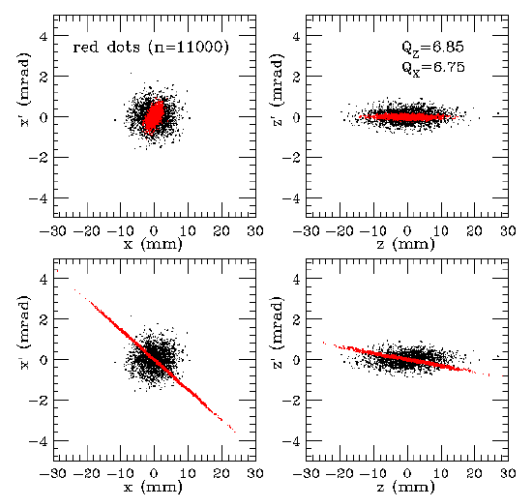


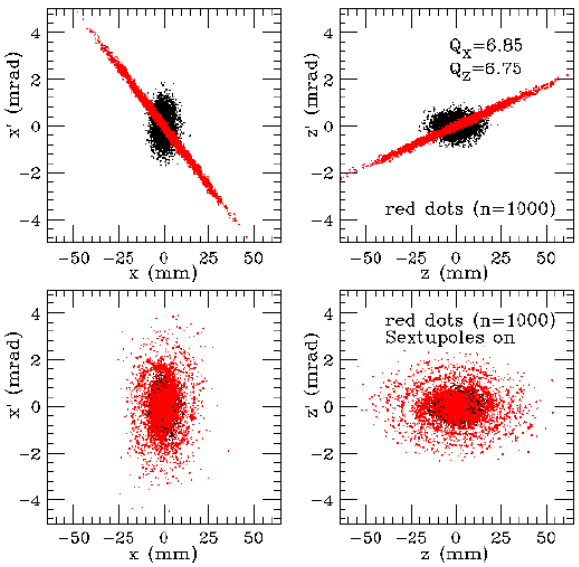
Effect of dipole field errors:

Red curves: Δx'=0.000020 rad, Δz'=0.000075 rad
Blue curves: No dipole error
Betatron tunes are regular operation tunes

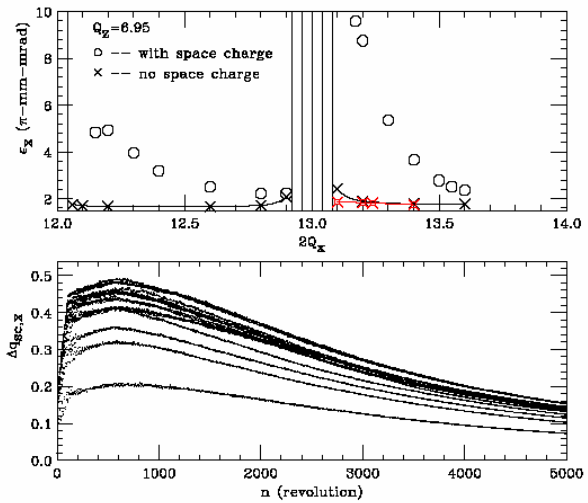


Effect of quadrupole field error:

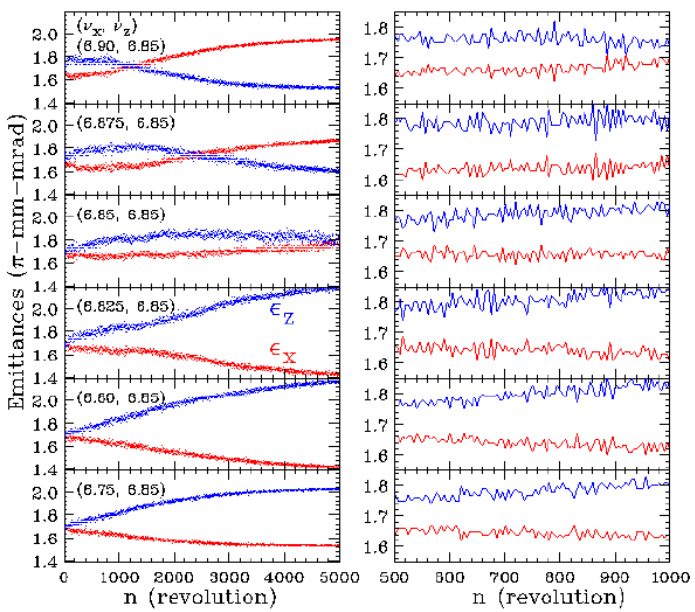




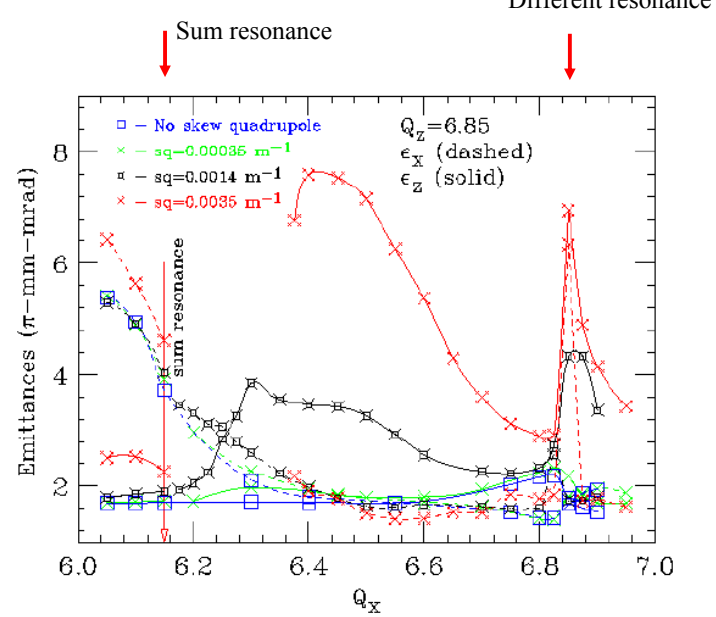
Effect of random quadrupole field error: 40e-4 m-1, while changing the horizontal betatron tune from 6 to 7.

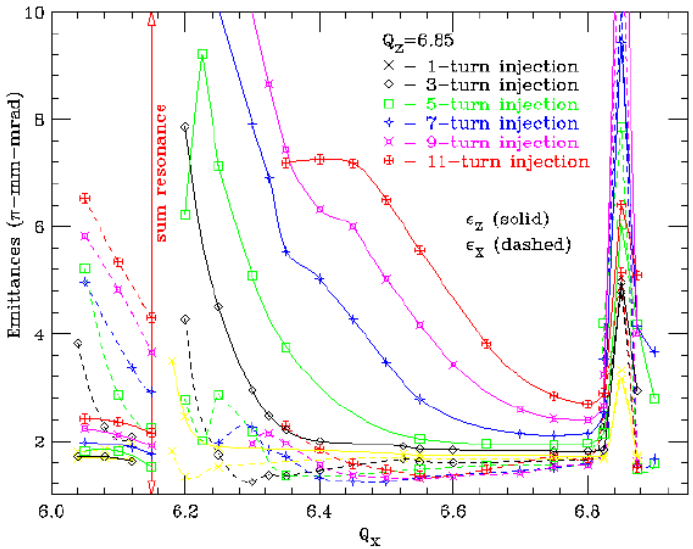


Montague Resonance: (No linear errors)

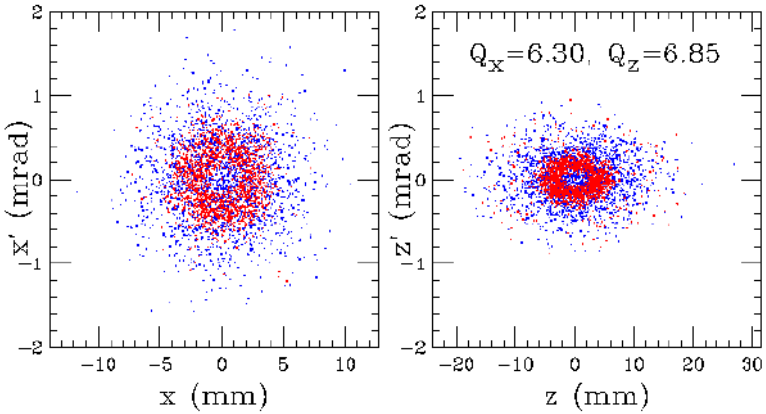


Effects of linear skew quadrupoles





Effect of a sum resonance!



Summaries:

1. We are able to fit experimental measured IPM data (vertical and horizontal) to deduce the betatron emittances of the beam. We note that the vertical emittance increases rapidly in the initial stage of acceleration. The horizontal emittance seems to be less affected by the space charge at the initial stage.
2. The bunch width oscillation after the transition crossing has been used to deduce the bunch-mismatch factor, and the result is consistent with the rf program in the Booster operation.
3. We use the rms space charge force model to carry out numerical simulations. We are able to fit the observed data of emittance growth. The emittance growth in the vertical plane has resulted mainly from the skew quadrupoles, that induce a sum resonance at Q_x+Q_z=integer. This induces emittance growth mainly in the vertical plane. The reason is that the Montague resonance suppresses the growth of the horizontal emittance, and enhance the vertical emittance by about 25%. The random dipole field error also generates about 25% vertical emittance growth.
4. **Need more work to understand more detailed mechanism of skew quadrupoles in the presence of Montague Resonance! A student is extending the simulation program using the error-function for the space charge kick model.**

4. What is the effect of the nonlinear systematic space charge resonances on beam emittances?

The space charge potential has the form of $\exp(-(x^2+z^2)/4\sigma^2)$. We know that the Montque resonance is produced by the x^2z^2 term in the potential. How about the systematic resonance induced by the terms x^4, z^4, x^2z^2, x^6, x^4z^2, x^2z^4, z^6, etc? Since the space charge potential follows the beam profile, which has the same superperiodicity, systematic resonances are located at 4qx=P, 4qz=P, 2qx+2qz=P, 6qx=P, 6qz=P, etc.

What is the effects of systematic resonances?

1) S. Machida, NIMA 384, 316 (1997)
2) Ingo Hofmann, Giuliano Franchetti, and Alexei V. Fedotov , HB2002, AIP conference proceedings
3) S. Igarashi et al., observed at the KEKPS at injection, PAC2003, p.2610 (2003)
4) Oliver Boine-Frankenheim observed the 4th order resonance in bunch rotation at the SIS18 simulations.

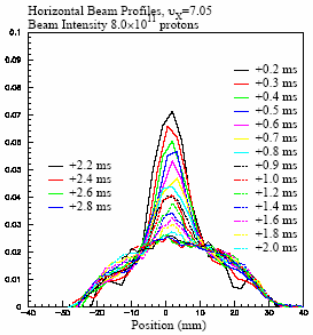
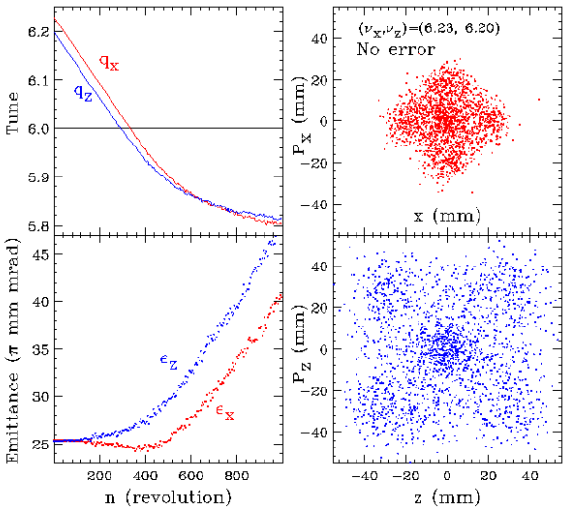


Figure 1: Horizontal beam profiles 0.2 ~ 2.8 ms after injection when the horizontal tune was 7.05 and the injection beam intensity was 8.0×10^{11} protons.

If one chooses the bare tunes at (6.23, 6.20), and 1000 injection-turns, with a total tune-shift of 1.1.
Note that the tune shift of SNS is only 0.15!

Phase space map at the end of injection.

The tune for small amplitude particles continue to decrease as the particle is injected!



$$V(x,z) = \frac{K_{sc}}{2}\int_0^\infty \frac{-1+\exp\{-\frac{x^2}{2\sigma_x^2+t}-\frac{z^2}{2\sigma_z^2+t}\}}{\sqrt{(2\sigma_x^2+t)(2\sigma_z^2+t)}}dt$$
$$\approx -\frac{K_{sc}}{2}\left\{\left(\frac{x^2}{\sigma_x(\sigma_x+\sigma_z)}+\frac{z^2}{\sigma_z(\sigma_x+\sigma_z)}\right)\right.$$
$$\left.-\frac{1}{8\sigma_x^2(\sigma_x+\sigma_z)^2}\left(\frac{2+R}{3}x^4+\frac{2}{R}x^2z^2+\frac{1+2R}{3R^3}z^4\right)+\cdots\right\}$$
$$V_{sc,4}(x,z) = -\frac{K_{sc}}{16\sigma_x^2(\sigma_x+\sigma_z)^2}\left(\frac{2+R}{3}x^4+\frac{2}{R}x^2z^2+\frac{1+2R}{3R^3}z^4\right)$$

$$V_{sc,4}(J_x,J_z,\psi_x,\psi_z,\theta) \approx -\sum|G_{4,0,\ell}|J_x^2\cos(4\psi_x-\ell\theta+\chi_{4,0,\ell})$$
$$-\sum|G_{0,4,\ell}|J_z^2\cos(4\psi_z-\ell\theta+\chi_{0,4,\ell})$$
$$-\sum|G_{2,2,\ell}|J_xJ_z\cos(2\psi_x+2\psi_z-\ell\theta+\chi_{2,2,\ell})$$
$$-\sum|G_{2,-2,\ell}|J_xJ_z\cos(2\psi_x-2\psi_z-\ell\theta+\chi_{2,-2,\ell}),$$

$$G_{4,0,\ell} = \frac{1}{2\pi}\oint\frac{K_{sc}\beta_x^2(2\sigma_x+\sigma_z)}{48\sigma_x^3(\sigma_x+\sigma_z)^2}\exp\{j(4\phi_x-4\nu_x\theta+\ell\theta)\}ds,$$
$$G_{0,4,\ell} = \frac{1}{2\pi}\oint\frac{K_{sc}\beta_z^2(\sigma_x+2\sigma_z)}{48\sigma_z^3(\sigma_x+\sigma_z)^2}\exp\{j(4\phi_z-4\nu_z\theta+\ell\theta)\}ds,$$
$$G_{2,2,\ell} = \frac{1}{2\pi}\oint\frac{K_{sc}\beta_x\beta_z}{16\sigma_x\sigma_z(\sigma_x+\sigma_z)^2}\exp\{j(2\phi_x+2\phi_z-2\nu_x\theta-2\nu_z\theta+\ell\theta)\}ds,$$
$$G_{2,-2,\ell} = \frac{1}{2\pi}\oint\frac{K_{sc}\beta_x\beta_z}{16\sigma_x\sigma_z(\sigma_x+\sigma_z)^2}\exp\{j(2\phi_x-2\phi_z-2\nu_x\theta+2\nu_z\theta+\ell\theta)\}ds,$$

FODO cells

$$\phi_x = \int_0^s\frac{ds}{\beta_x},\quad \phi_z = \int_0^s\frac{ds}{\beta_z}.$$

$$G_{4,0,\ell} = 1.48\times10^{-2}(K_{sc}C/8\pi\epsilon^2)$$
$$G_{0,4,\ell} = -1.49\times10^{-2}(K_{sc}C/8\pi\epsilon^2)$$
$$G_{2,2,\ell} = -1.04\times10^{-4}(K_{sc}C/8\pi\epsilon^2)$$
$$G_{2,-2,\ell} = 1.17\times10^{-1}(K_{sc}C/8\pi\epsilon^2)$$

2-kick approximation: $0.00702, -0.00604, -0.000576,$ and $0.0982.$

See also: S Machida, Nucl. Inst. Methods A384, 316 (1997)

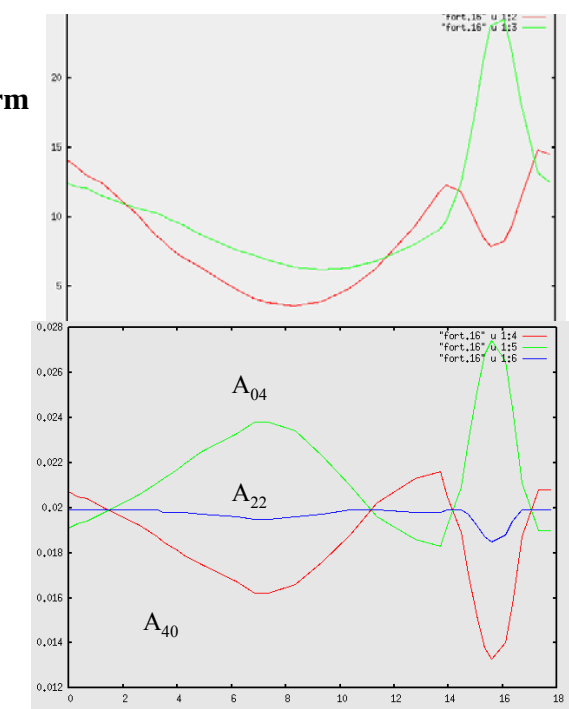
SIS18
space charge octupole term

$G_{4,0,\ell}$=1.25E-3

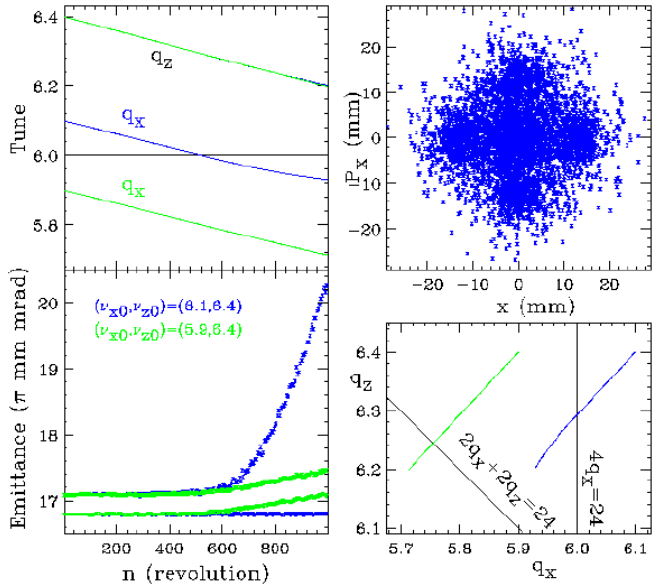
$G_{0,4,\ell}$=1.33E-3

$G_{2,2,\ell}$=2.62E-4

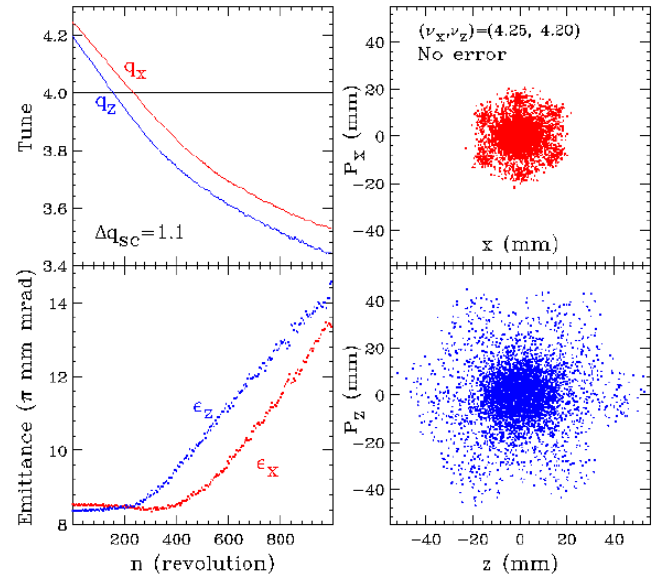
$G_{2,-2,0}$=0.124



Effect of 4qx=24, and 2qx+2qz=24:



How about the 6qx=P and 6qz=P resonances?



5. Non-scaling FFAG

Thanks to Shinji's talk at GSI!

design by A. Ruggiero, AP-Technote: 219 (BNL-ADD)

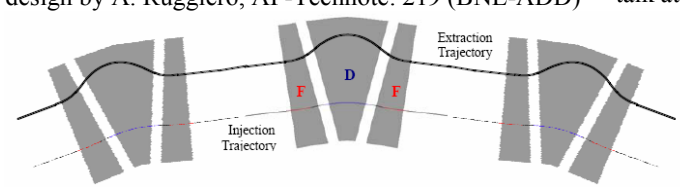


Table 1. Major Parameters of the 3 FFAG Rings (Proton Driver)

		Inj. Ring	*LE Ring*	*HE Ring*
Energy: Inj.	GeV	0.40	1.50	4.45
Ext.		1.50	4.45	11.6
β Inj.		0.7131	0.9230	0.9847
Ext.		0.9230	0.9847	0.9972
Δp/p	±%	40.45	40.43	40.41
Circumference	m	807.091	818.960	830.829
No. of Periods		136	136	136
Period Length	m	5.934	6.022	6.109
Harmonic No.		136	138	140
RF Inj.	MHz	36.02	46.03	49.75
Ext.	λ = 5.9345 m	46.03	49.75	50.38

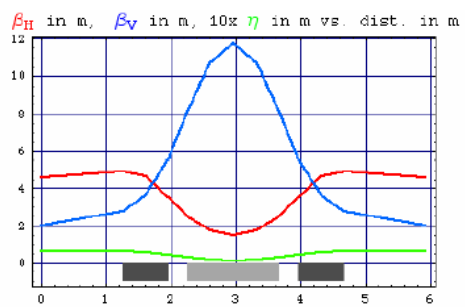


Figure 4. Lattice Functions along the Length of a Period

Table 3. Global Lattice Parameters

Phase Adv. / Cell	H	105.234°
	V	99.9395°
Betatron Tune,	H	39.755
	V	37.755
Nat. Chromaticity,	H	-0.9263
	V	-1.8052
Transition Energy,	γ_T	105.482 i

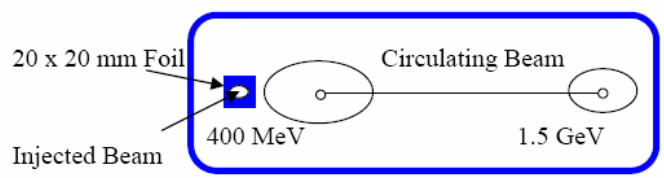


Figure 5. Beam Footprint in the Injection area with a 30 x 10 cm Vacuum Chamber

Table 5. Space-Charge, Beam Size and Beam Intensity

		Inj. Ring	*LE Ring*	*HE Ring*
Protons / pulse		1.0×10^{14}	1.0×10^{14}	1.0×10^{14}
Average Beam Current	mA	1.60	1.60	1.60
Average Beam Power	MW	2.40	7.12	18.56
Full Nor. Emittance	π mm-mrad	100	100	100
Actual Inj. Emittance	π mm-mrad	98.32	41.69	17.68
Bunching Factor		4.0	4.0	4.0
Tune-Shift		0.343	0.188	0.085
Half Vert. Beam Size	cm	2.12	1.38	0.90
Half Hor. Beam Size	cm	3.41	2.22	1.46

Table 6. RF Acceleration Parameters for the Pulsed Mode

		Inj. Ring	*LE Ring*	*HE Ring*
Energy Gain per Turn	MeV/turn	0.60	0.90	2.00
No. of Revolutions		1834	3278	3576
RF Peak Voltage	MVolt	1.20	1.80	4.00
Acceleration Period	ms	6.137	9.398	10.001
Injection Period	ms	1.144	--	--
Max. Repetition Rate	kHz	0.137	0.106	0.100
Gap Voltage	kVolt	20	30	40
Gaps / Cavity		2	2	2
Number of Cavities		30	30	50

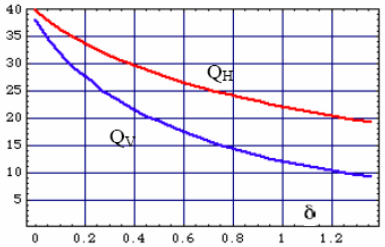


Figure 10. Tune Variation vs. Momentum

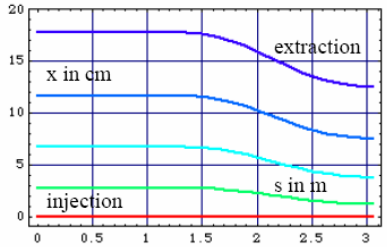
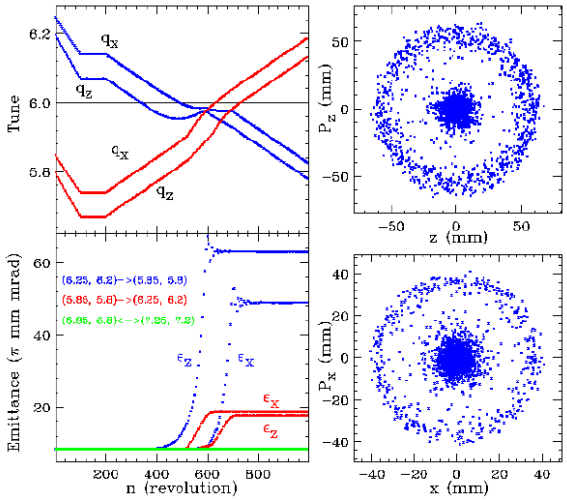
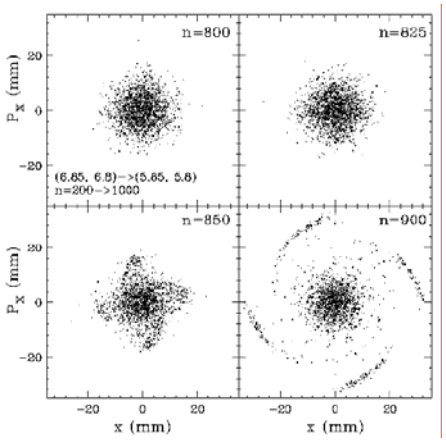
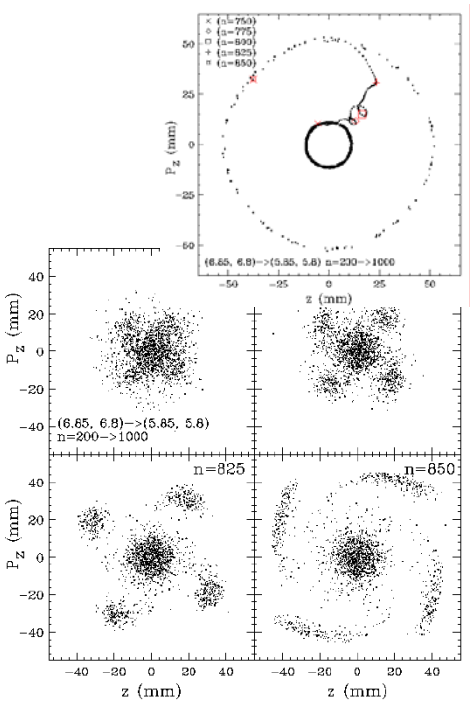


Figure 11. Closed Orbits vs Half-Period Length

dQx/dn=20/2000=0.01
dQz/dn=30/2000=0.015



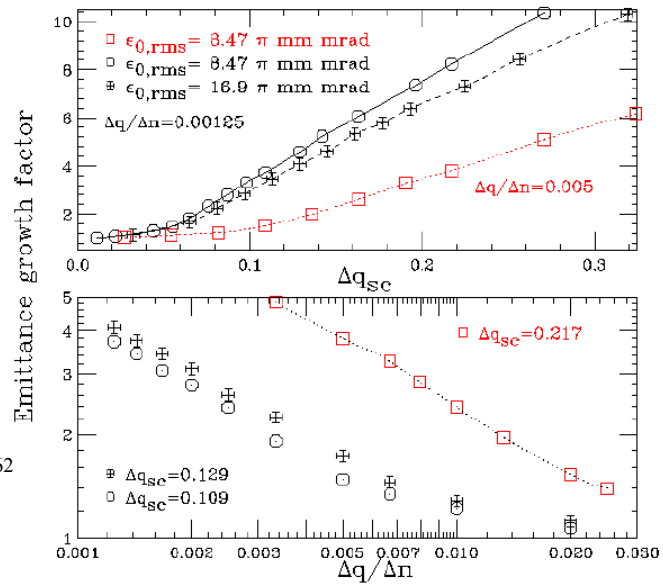




EGF=ε(final)/ε(initial)

EGF~($\Delta\nu_{sc}$)

EGF~$(dq/dn)^{-0.62}$



Gradient errors: $y'' + K(s)y + k(s)y = 0$

Floquet transformation: $\eta = \frac{y}{\sqrt{\beta_y}}, \quad \varphi = \frac{1}{\nu}\int_0^s \frac{ds}{\beta_y}$ $\qquad \ddot{\eta} + \nu^2\eta + \nu^2\beta_y^2 k(s)\eta = 0$

$$\boxed{\nu\beta_y^2 k(s) = \sum J_p e^{jp\varphi}, \quad J_p = \frac{1}{2\pi}\oint \nu\beta_y^2 k(s) e^{-jp\varphi} d\varphi = \frac{1}{2\pi}\oint \beta_y k(s) e^{-jp\varphi} ds}$$

$$\ddot{\eta} + \nu^2\eta + \nu\sum J_p e^{jp\varphi}\eta \approx \boxed{\ddot{\eta} + \nu^2\eta + 2\nu g_p \cos(p\varphi + \chi_p)\eta = 0}$$

$$H = \tfrac{1}{2}\dot{\eta}^2 + \tfrac{1}{2}\nu^2\eta^2 + \nu g_p \cos(p\varphi + \chi_p)\eta^2$$

Mathieu's equation

$$\eta = \frac{\sqrt{2J}}{\sqrt{\nu}}\cos\psi,$$

$$H = \nu J + J g_p \cos(p\varphi + \chi_p)(1 + \cos 2\psi) \approx \nu J + \tfrac{1}{2} J g_p \cos(2\psi - p\varphi - \chi_p)$$

$$F_2 = (\psi - \tfrac{1}{2}p\varphi - \tfrac{1}{2}\chi_p)I \qquad H = (\nu - \tfrac{1}{2}p)I + \tfrac{1}{2} I g_p \cos(2\phi)$$

$$\ddot{I} = g_p^2 I + 2[g_p(\nu - \tfrac{1}{2}p)\cos 2\phi]I$$

At ν=p/2: $\quad I = ae^{g_p\varphi} + be^{-g_p\varphi} \sim e^{2\pi g_p n}$

Sum resonances driven by the skew quads:

$$x'' + K_x(s)x + 2gz' - (q - g')z = 0,$$
$$z'' + K_z(s)z + 2gx' - (q + g')x = 0,$$

$$G_{1,\mp 1,\ell}\, e^{j\chi_\mp} = \frac{1}{2\pi}\oint \sqrt{\beta_x\beta_z}\, A_{lc\mp}(s)\, e^{j[\chi_x \mp \chi_z - (\nu_x \mp \nu_z - \ell)\theta]} ds.$$

$$A_{lc\mp}(s) = -\frac{a_1}{\rho} + g(s)\left(\frac{\alpha_x}{\beta_x} - \frac{\alpha_z}{\beta_z}\right) + jg(s)\left(\frac{1}{\beta_x} \pm \frac{1}{\beta_z}\right),$$

$$H = \nu_x J_x + \nu_z J_z + g\sqrt{J_x J_z}\cos(\psi_x + \psi_z - \ell\theta + \chi), \qquad g = |G_{1,1,\ell}|$$

$$F_2 = (\psi_x + \psi_z - \ell\theta + \chi)I_1 + \psi_z I_2 \qquad I_1 = J_x, \quad I_2 = J_z - J_x$$

$$H = \delta I_1 + g\sqrt{I_1(I_1 + I_2)}\cos\phi_1 + \nu_z I_2, \qquad \delta = (\nu_x + \nu_z - \ell)$$

$$\ddot{I}_1 = [g^2 - \delta^2]I_1 + (\tfrac{1}{2}g^2 I_2 + \delta H)$$

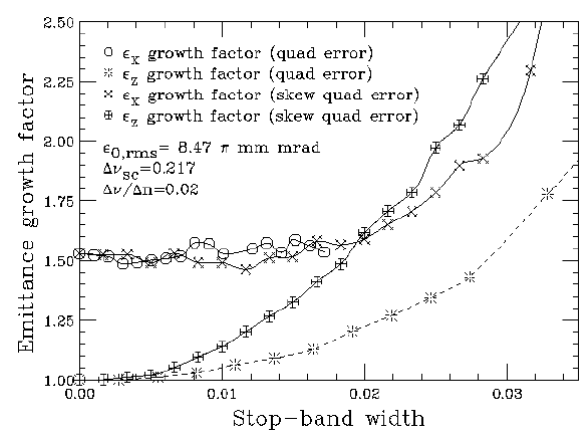
$$I_1 = ae^{\sqrt{g^2-\delta^2}\theta} + be^{-\sqrt{g^2-\delta^2}\theta} + c + d\theta + \tfrac{1}{2}(\tfrac{1}{2}g^2 I_2 + \delta H)\theta^2$$

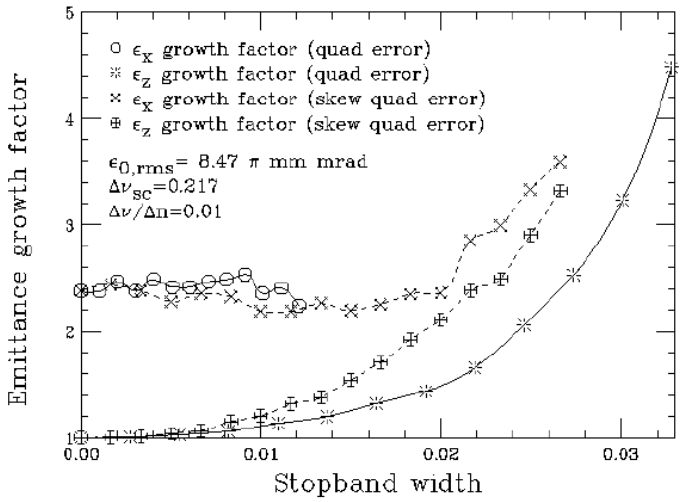
$$\sim ae^{2\pi\sqrt{g^2-\delta^2}n}$$

When the tunes ramp through resonances, the number of turns that the tune stays on resonance is Δn=g/(dν/dn). Thus we expect that the emittance growth is given by

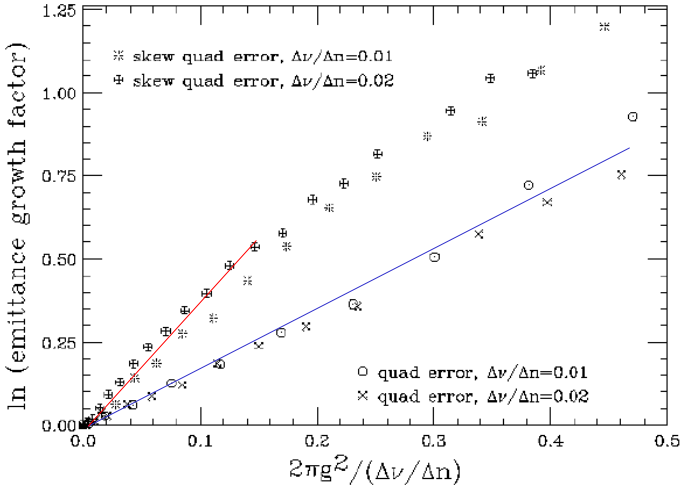
$$\mathrm{EGF} = \exp\left[\lambda \frac{2\pi g^2}{d\nu/dn}\right]$$

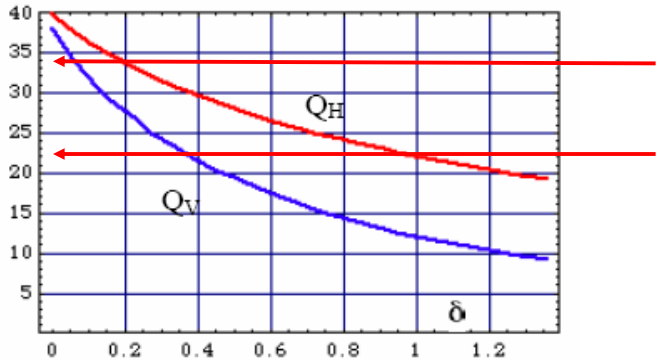
In this calculation, the tunes are ramped from (6.85,7.80) to (5.85, 6.80)





λ≈1.5 for quadrupole error, and 3.5 for skew quadrupole error





136/4=34
136/6=22.6666

Figure 10. Tune Variation vs. Momentum

Conclusion:

1. Systematic Nonlinear space charge resonances can be important in high intensity accelerators
2. For future neutron source design, one should try to avoid the systematic nonlinear space charge resonance, if it is possible!
3. For the non-scaling FFAG, the nonlinear resonances induced by the space charge potential can be the limiting factor. These resonances limit the phase advance of each basic cell to within π/2 to π/3, and thus the momentum acceptance is highly constrained.
4. I also find that the emittance growth factor for quadrupole and skew-quadrupole errors obeys a simple scaling law:

$$\mathrm{EGF} = \exp\left[\frac{\lambda 2\pi g^2}{d\nu / dn}\right]$$