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Technical Note

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SIMPLE APPROXIMATION FOR SYNCHROTRON FREQUENCY

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Abstract

The synchrotron frequency distribution for particles within the stationary bucket can be approximated with good accuracy by the formula

$$\Omega(\phi_o) = \omega_o \sqrt{1 - \left(\frac{\phi_o}{\pi}\right)^2} ,$$

where ω_o is the usual synchrotron frequency for small amplitude, ϕ_o is the particle amplitude angle.

1. Linear-Parametric Approximation

Within a stationary bucket in longitudinal phase space, each particle exercises a synchrotron oscillation with its own constant synchrotron frequency, which is higher toward the bucket center and lower for particles close to the separatrix. I will apply a term central (synchrotron) frequency for the particles of infinitesimal amplitude. That is what is usually called synchrotron frequency, because it comes from the equation

$$\ddot{\phi} + \omega_o^2 \phi = 0 , \tag{1}$$

approximating synchrotron oscillations of particles with such a small angular amplitude $\phi << 1$, that satisfy

$$\phi = \mathrm{Sin}\ \phi . \tag{2}$$

For arbitrary ϕ, however, the approximate equation (1) is replaced by an exact one

$$\ddot{\phi} + \omega_o^2 \operatorname{Sin} \phi = 0, \qquad -\pi \leq \phi \leq \pi \tag{3}$$

whose solution $\phi = \phi(t)$ should satisfy the initial conditions*

$$\phi(0) = \phi_o \ , \quad \dot{\phi}(0) = 0 \ . \tag{4}$$

An approximation (2) is a part of a simple class of linear-parametric functions defined by

$$\operatorname{Sin} \phi \simeq \phi p^2(\phi_o) \tag{5}$$

and applicable to the problem (3), (4).

After the substitution of (5) to (3), the approximate synchrotron frequency

$$\Omega_p(\phi_o) = \omega_o p(\phi_o) \tag{6}$$

should be compared with the exact one $\Omega = \Omega(\phi_o) = \omega_o F(\phi_o)$. The latter can be found by the use of an elliptic integral[1] or by tracking the particle motion numerically. I did the tracking. In the next section, we will compare several approximating functions $p(\phi_o)$.

2. Comparison of Approximating Functions

Figures 1 to 4 show the computed results for four approximating functions defined as follows:

$$p_2^2(\phi_o) = 1 - \frac{\phi_o^2}{3!} \ , \tag{7}$$

*Any other ($\dot{\phi}(0) \neq 0$) initial conditions can be reduced to (3) by the appropriate shifting of the time reference frame: $t \rightarrow t + t_o$.

$$p_3^2(\phi_o) = 1 - \frac{\phi_o^2}{3!} + \frac{\phi_o^4}{5!} \quad , \tag{8}$$

$$p_s^2(\phi_o) = \frac{\text{Sin } \phi_o}{\phi_o} \quad , \tag{9}$$

$$p_\pi^2 = 1 - \left(\frac{\phi_o}{\pi}\right)^2 \quad . \tag{10}$$

All of the above originated from linear-parametric representation (5).

Let us now turn to the figures showing approximations and their errors. Each figure has three curves. Two of them are $F(\phi_o)$ (exact distribution) and $p_a(\phi_o)$ (approximate distribution ($a = 2, 3, s, \pi$) both starting at $\phi_o = 0$, $F(0) = p_a(0) = 1$. The space between those two curves is shaded. The third curve $E(\phi_o)$, starting at $\phi_o = 0$, $E(0) = 0$ is relative error:

$$E(\phi_o) = \frac{F(\phi_o) - p_a(\phi_o)}{F(\phi_o)} \quad . \tag{11}$$

The worst approximation is the first which comes from obvious Taylor expansion up to the second term. Because this approximation is valid only for the short interval of argument ϕ_o, Figure 1 is not completed for ϕ_o close to π.

Figure 2 represents the approximation coming from the Taylor expansion up to the third term. The improvement in accuracy is not very dramatic for the price of increased complexity in p_3.

Figure 3 shows that trying to avoid Taylor expansion does not pay in accuracy. Maybe the only profit from this approximation is the lower boundary estimation

$$\frac{\text{Sin } \phi_o}{\phi_o} \leq F^2(\phi_o) \quad , \qquad 0 \leq \phi_o \leq \pi \quad . \tag{12}$$

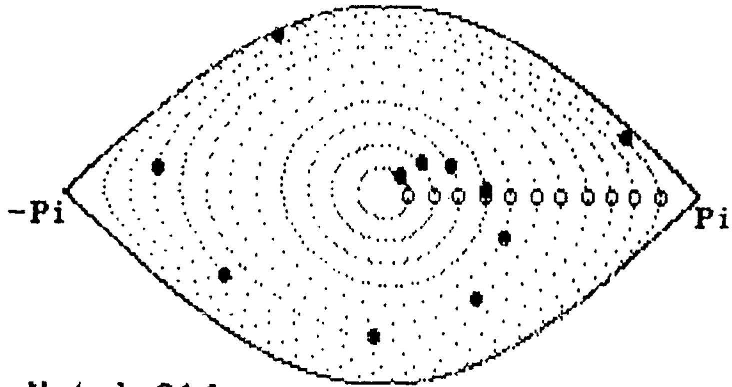

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Figure 4 shows unexpectedly that a small correction in the second order Taylor expansion pays off very well: the relative error is less than 5% for at least 80% of the argument region.

It is interesting to compare the exact distribution $F(\phi_o)$ with its own expansion[1] up to the second term:

$$F(\phi_o) \simeq p_E(\phi_o) = \left(1 + \frac{\phi_o^2}{16}\right)^{-1} . \qquad (13)$$

The comparison of F and p_E is shown in Figure 5. One can see P_E is good only for 60% of the region.

So, we choose as an approximate synchrotron frequency distribution

$$\Omega(\phi_o) = \omega_o \sqrt{1 - \left(\frac{\phi_o}{\pi}\right)^2} . \qquad (14)$$

Try it--you'll like it!

<u>Reference</u>

1. L.D. Landau and E.M. Lifshitz, Mechanics, Third Ed., 1982, Pergamon Press, Para. 11, Problem 1.

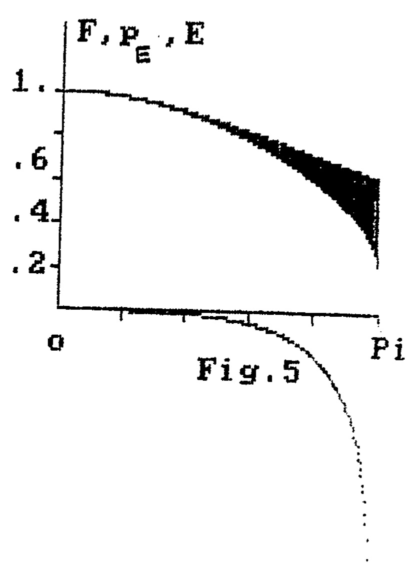

Fig.5

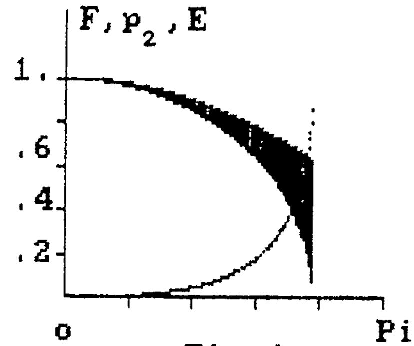

Fig.1

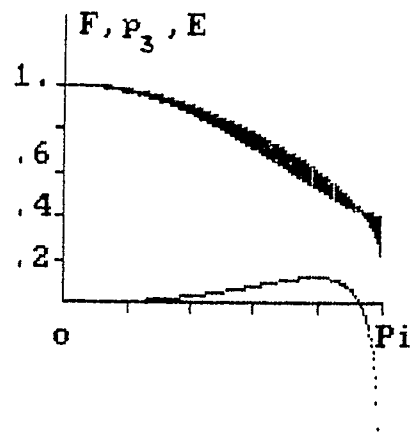

Fig.2

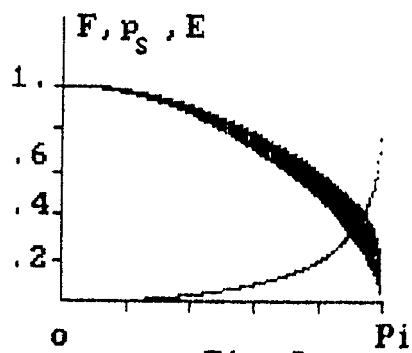

Fig.3

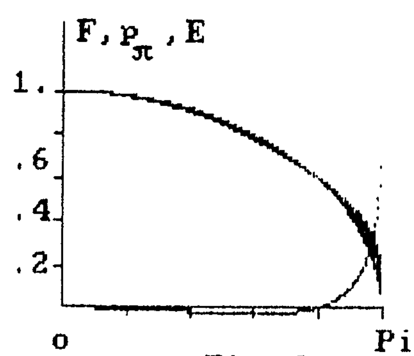

Fig.4